MATH 200, Lec 2 Solutions to Midterm 1

1. (a) [4 points] Write a statement that is logically equivalent to the one below, but uses no negatives.

   If you didn't do the homework, then you won't pass the exam.

   Solution: This is an implication of the form $not\ P \implies not\ Q$, where the statement P is "You did the homework", and the statement Q is "you pass the exam". The contrapositive (which is always an equivalent statement) of $not\ P \implies not\ Q$ is $Q \implies P$, that is,

   If you pass the exam, then you did the homework.

   Note that the statement "If you did the homework, you will pass the exam." is not equivalent to the original. Rather, it is the converse.

   (b) [4 points] Write the negation of the statement below, using no negatives:

   For every positive real number ε and for every integer x, there is an integer y so that

$$0 \leq \frac{x}{y} \quad \textit{and} \quad \frac{x}{y} < \varepsilon$$

   Solution: Some people found this easier to do by first writing the original symbolically, which is

$$\forall \varepsilon \in \mathbb{R}^+\ \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, 0 \leq \frac{x}{y} \text{ and } \frac{x}{y} < \varepsilon$$

   To negate such a statement, we exchange the quantifiers $\forall$ and $\exists$ and negate what follows, giving us

$$\exists \varepsilon \in \mathbb{R}^+\ \exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \ \text{not} \left(0 \leq \frac{x}{y} \text{ and } \frac{x}{y} < \varepsilon\right)$$

   Now we write the negation of the innermost part. Recall that not(A and B) is (not A) or (not B), so we have

$$\exists \varepsilon \in \mathbb{R}^+\ \exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \ 0 > \frac{x}{y} \text{ or } \frac{x}{y} \geq \varepsilon$$

   In words, we would say this as

   There is a positive real number ε and an integer x, so that for any integer y we have either

$$\frac{x}{y} < 0 \quad \textit{or} \quad \frac{x}{y} \geq \varepsilon$$

2. 8 points Prove that for any integer n, if n^2 is odd, then n is odd.

Solution: It is most straightforward to prove the contrapositive, that is, to show that if n is even, then n^2 is also even.

If n is even, then there is an integer q so that $n = 2q$. Then $n^2 = (2q)^2 = 2(2q^2)$. We have shown there is an integer m (namely, $m = 2q^2$) so that $n^2 = 2m$, so n^2 is even, as desired.

3. 6 points Prove that for any sets A, B, and C, $(A \cap C) - B = (A - B) \cap C$

Solution: This can be done in several essentially equivalent ways. The simplest is to note that for any sets S and R, $S - R = S \cap R^c$ (where R^c is the complement of R. Then we have

$$(A \cap C) - B = (A \cap C) \cap B^c = A \cap B^c \cap C = (A - B) \cap C.$$

Another way is to take an element of one set and argue that it lies in the other, and vice-versa. Even though it is essentially equivalent, I'll do that too:

Suppose $x \in (A \cap C) - B$. This means that $x \in A \cap C$ and $x \notin B$. Since $x \in (A \cap C)$, we have $x \in A$ and $x \in C$. Reordering, we have $x \in A$ and $x \notin B$ and $x \in C$. Putting these together gives us $x \in (A - B) \cap C$, which shows $(A \cap C) - B \subseteq (A - B) \cap C$. The argument above is completely reversible, so we also know $(A - B) \cap C \subseteq (A \cap C) - B$, giving the desired result.

Many students chose to do this via a truth table with 8 lines:

$x \in A$	$x \in B$	$x \in C$	$x \in A \cap C$	$x \in (A \cap C) - B$	$x \in A - B$	$x \in (A - B) \cap C$
T	T	T	T	F	F	F
T	T	F	F	F	F	F
T	**F**	**T**	**T**	**T**	**T**	**T**
T	F	F	F	F	T	F
F	T	T	F	F	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Note that values for membership in both sets agree; specifically, both are false except in the third line of the table.

Finally, a Venn diagram would be acceptable, provided that the regions $A \cap C$ and $A - B$ are indicated as well as $(A \cap C) - B$ and $(A - B) \cap C$ (the last two are of course the same).

4. 8 points Prove that for any positive integer n, $4^n + 5$ is divisible by 3. You might find induction helpful. Recall that $4 = 3 + 1$.

Solution: We'll do this by induction.

For the base case, notice that if $n = 1$ we have $4^1 + 5 = 9$, and 9 is divisible by 3.

Now we show that whenever $4^k + 5$ is divisible by 3, we must also have $4^{k+1} + 5$ divisible by 3. To see this, notice that

$$4^{k+1} + 5 = (3 + 1) \cdot 4^k + 5 = 3 \cdot 4^k + (4^k + 5)$$

Since $4^k + 5$ is divisible by 3 by our inductive hypothesis, there is some integer q so that $4^k + 5 = 3q$. This means we have shown

$$4^{k+1} + 5 = 3 \cdot 4^k + 3q = 3(4^k + q),$$

giving the desired conclusion.

5. Indicate whether each of the following statements is true or false, and justify your answer with a proof.

(a) 3 points $\forall x \in \mathbb{R}, \exists y \in \mathbb{R},\ x + y > 0$ True False

Solution: True. We must show that for any given real number x, we can find a y so that $x + y$ is positive. Choosing $y = 1 - x$ works fine, since $x + (1 - x) = 1$ and $1 > 0$. Of course, there are plenty of other choices that work just as well.

(b) 3 points $\exists y \in \mathbb{R}, \forall x \in \mathbb{R},\ x + y > 0$ True False

Solution: False. Suppose there were such a value of y; let's call it Q. Then it would be true that for any choice of $x \in \mathbb{R}$, $x + Q$ is positive. If we take $x = -Q$, this fails. So no such Q can exist.

An alternative is to prove the negation of the statement is true. That is, we can show that $\forall y \in \mathbb{R}, \exists x \in \mathbb{R},\ x + y \leq 0$. But this is almost the same as the answer to the previous part: given any such y, let $x = -1 - y$, and then $x + y = -1$. Since the negation of the statement is true, the original statement must be false.

(c) 3 points $\exists x \in \mathbb{R}, \forall y \in \mathbb{R},\ xy \geq 0$ True False

Solution: True. Note that if $x = 0$, then no matter what y is, we have $xy = 0 \cdot y = 0$, as desired.

6. Let $f : \mathbb{R} \to \mathbb{R}^2$ be given by $f(x) = (x + 1, x^2 + 1)$.

   (a) [4 points] Is f surjective? Prove or disprove your answer.

   Solution: f is not surjective.

   If it were, then for any ordered pair $(a, b) \in \mathbb{R}^2$, we could find x so that $f(x) = (a, b)$. But there is no x so that $f(x) = (1, 0)$. If there were, then since $x + 1 = 1$, we'd have $x = 0$. But $f(0) = (1, 1) \neq (1, 0)$.

   (b) [4 points] Is f injective? Prove or disprove your answer.

   Solution: Yes, f is injective.

   To see this, suppose $f(x) = f(y)$. Then we have $(x + 1, x^2 + 1) = (y + 1, y^2 + 1)$, and in particular, $x + 1 = y + 1$. This means $x = y$, and so f is an injection.