MATH 200 Solutions to Midterm 1

4 pts. 1. (a) Write a statement that is logically equivalent to the one below, but uses no negatives.

If an integer is not odd, then it cannot be written as the product of two odd integers.

Solution: The contrapositive of the above statement is equivalent. This is:
If an integer can be written as the product of two odd integers, then that integer is odd.

6 pts. (b) Let $S \subseteq \mathbb{R}$. Write the negation of the statement below, using no negatives:

There exists a positive real number M so that if $x \in S$, then $-M < x < M$.

Give an example of a set of real numbers for which the property above fails to hold.

Solution: The negation of the statement is
For any positive real number M, there is an $x \in S$ for which either $x \leq -M$ or $x \geq M$.
There are plenty of examples of such sets: $\mathbb{R}$ is one, as is $\mathbb{R}^+$, or $\mathbb{Z}$, or any number of others.

8 pts. 2. For any sets A, B, and C, prove that

if $A \cap B \subseteq C$ and $x \in B$, then $x \notin A - C$.

Feel free to use Venn diagrams to illustrate your proof, but a proof by Venn diagrams alone will not be given full credit.

Solution: We prove this by contradiction. Our hypotheses is $A \cap B \subseteq C$ and $x \in B$, and we also assume $x \in A - C$ for contradiciton. Thus $x \in A$. Since $x \in B$ as well, we have $x \in A \cap B$. Since every element of $A \cap B$ is also in C, we have $x \in C$, a contradiction.

8 pts. 3. Prove that for any real number $x \neq 1$ and any positive integer n,

$$\sum_{j=0}^{n} x^j = \frac{1 - x^{n+1}}{1 - x}.$$

Using induction on n will likely be helpful.

Solution: As the base case, take $n = 1$. Then we must establish that

$$1 + x = \frac{1 - x^2}{1 - x}$$

But this holds, since $1 - x^2 = (1 - x)(1 + x)$.

For the inductive step, we must show that if $\sum_{j=0}^{k} x^j = \frac{1-x^{k+1}}{1-x}$, then $\sum_{j=0}^{k+1} x^j = \frac{1-x^{k+2}}{1-x}$.

So, we begin with the left-hand-side, and manipulate it to arrive at the right.

$$\begin{aligned}
\sum_{j=0}^{k+1} x^j &= x^{k+1} + \sum_{j=0}^{k} x^j \\
&= x^{k+1} + \frac{1-x^{k+1}}{1-x} && \text{applying the inductive hypothesis} \\
&= \frac{x^{k+1}-x^{k+2}}{1-x} + \frac{1-x^{k+1}}{1-x} \\
&= \frac{1-x^{k+2}}{1-x}.
\end{aligned}$$

8 pts. 4. Let $f : X \to Y$ and $g : Y \to Z$ be injective functions. Is $h = g \circ f$ an injective function? Give a proof or counterexample.

Solution: Yes, $g \circ f$ is injective.

To see this, we must show that if $h(x_1) = h(x_2)$, then $x_1 = x_2$. So, suppose we have $h(x_1) = h(x_2)$, that is, $g(f(x_1)) = g(f(x_2))$. Now, since g is injective, we know that $f(x_1) = f(x_2)$, and since f is also injective, we know that $x_1 = x_2$, as desired.

8 pts. 5. Prove that for every pair of integers m and n, if $n - m$ is even, then $n^2 - m$ is also even.

(You may freely use the fact that the sum of two odd numbers is even, that the sum of an odd number and an even number is odd, the product of an odd and an even is even, etc.)

Solution: The simplest way to establish this is just to check the various possibilities.

n **even,** m **even.** The difference of two evens is even. Since n is even, so is n^2, and since m is even, $n^2 - m$ is even.

n **even,** m **odd.** Since $n - m$ is not even in this case, we need not consider further.

n **odd,** m **even.** Again, $n - m$ is odd, so we are done here, too.

n **odd,** m **odd.** The difference of two odd numbers is even, so we have to check that $n^2 - m$ is even. But since n is odd, so is n^2. An odd number minus another odd number is always even, and so $n^2 - m$ is even.

6. Indicate whether each of the following statements is true or false, and justify your answer with a proof. You may use the usual properties of real numbers, including those of inequalities.

3 pts. (a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R},\ 0 < y < x$ True False

Solution: False. If $x \leq 0$, the statement does not hold.

3 pts. (b) $\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R},\ 0 < y < x$ True False

Solution: True. Given any $x \in \mathbb{R}^+$, we can take $y = x/2$. Then $0 < y < x$.

3 pts. (c) $\exists y \in \mathbb{R}^+, \forall x \in \mathbb{R},\ 0 < y < x$ True False

Solution: False. If such a y existed, it would contradict the previous part (since we could let $x = y/2$.

6 pts. 7. (a) Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be given by $f(x, y) = (x + y^2, 2y)$.
Is f injective? surjective? bijective? Prove or disprove your answer.

Solution: This function is a bijection. To verify, we must show it is surjective and injective, or we can exhibit the inverse and note that it is a well-defined function. Let's do both:

Suppose $f(x, y) = (z, w)$. Then $w = 2y$, that is, $y = w/2$, and so $x = z - w^2/4$. Thus,

$$f^{-1}(z, w) = \left(z - \frac{w^2}{4}, \frac{w}{2}\right).$$

Since the inverse consists of polynomials, it is well-defined on all of $\mathbb{R} \times \mathbb{R}$, and so f is a bijection.

Alternatively (really, it is the same), f is surjective, since if (z, w) is any element of the co-domain,

$$f\left(z - \frac{w^2}{4}, \frac{w}{2}\right) = \left(z - \frac{w^2}{4} + \frac{w^2}{4}, 2 \cdot \frac{w}{2}\right) = (z, w).$$

Similarly, if $f(x_1, y_1) = f(x_2, y_2)$, then $2y_1 = 2y_2$ and so $y_1 = y_2$. But then since $x_1 - y_1^2 = x_2 - y_2^2$, we have $x_1 = x_2$. Hence f is an injection.

6 pts. (b) Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be given by $f(x, y) = (x + y^2, 2y)$.
Is f injective? surjective? bijective? Prove or disprove your answer.

Solution: The above argument for injectivity works whether x and y are integers or real numbers, so f is injective.

However, f is not surjective. There is no pair of integers (x, y) so that $f(x, y) = (0, 1)$. If so, we would need an integer y so that $2y = 1$, and there is no such integer.

Since f is not surjective, it is not bijective.