Solutions to Second Midterm

1. *(4 points)* Let A be a set with a elements, B be a set with b elements, and C be a set with c elements. Then the cardinality of $A \times (B \cup C)$ is $a(b+c)$.

Solution: This is a **False** statement, since B and C may have some elements in common, meaning that the cardinality of $B \cup C$ is at most $b+c$, not equal. The true statment is that the cardinality of $A \times (B \cup C)$ is no larger than $a(b+c)$.

2. *(16 points)* Prove that $\triangle ABC$ is equilateral (that is, $|AB| = |BC| = |AC|$) if and only if $\triangle ABC$ is equiangular ($m\angle A = m\angle B = m\angle C$).

Solution: First, we assume $\triangle ABC$ is equiangular and show it is equilateral. Since $m\angle B = m\angle C$, by the base angles equal theorem, $\triangle ABC$ is isosceles with apex A, so $|AB| = |AC|$. Also since Since $m\angle B = m\angle A$, $\triangle ABC$ is isosceles with apex C, and hence $|AC| = |BC|$. Thus $|AB| = |BC| = |AC|$ by transitivity, so $\triangle ABC$ is equilateral.

For the other direction, we assume $|AB| = |BC| = |AC|$ and show that the three angles are equal. This is done in an analagous manner, using the other direction of the base angles equal theorem. That is, since $|AB| = |AC|$, we have $m\angle C = m\angle B$. Similarly, since $|BC| = |BA|$, we get $m\angle C = m\angle A$. Thus, the triangle is equiangular.

3. *(16 points)* Is the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (x^2 - 1 + y, x)$ a bijection? Give a proof of your answer.

Solution: To show that f is a bijection, we must show it is one-to-one and onto.

The simplest way to do this is to calculate the inverse function, and show that it is a well-defined function (which says that f is one-to-one) defined on the whole domain (which says f is onto).

We let $(z,w) = f(x,y) = (x^2 - 1 + y, x)$, and solve for x and y. Then $w = x$ and $z = x^2 - 1 + y$, and substituting the first into the second, we get $z = w^2 - 1 + y$, so $y = z - w^2 + 1$. That is, the inverse function of f is given by

$$f^{-1}(z,w) = (w, z - w^2 + 1)$$

This is a well-defined function, since whenever we put in a pair (z,w), we get a well-defined output. It is also defined on the whole domain: there are no restrictions on z and w, because the coordinates of the function are just polynomials in z and w.

While that is the simplest way, let's do it the other way, as well. This comes down to the same thing, but here it is.

1-1: To show that f is 1-1, we must show that whenever $f(x,y) = f(a,b)$, then necessarily $(x,y) = (a,b)$. So suppose $f(x,y) = f(a,b)$, that is, $(x^2 - 1 + y, x) = (a^2 - 1 + b, a)$. So $x = a$ and $x^2 - 1 + y = a^2 - 1 + b$. Substituting, we get $a^2 - 1 + y = a^2 - 1 + b$, so $b = a$. Thus $(x,y) = (a,b)$, and f is 1-1.

onto: To show that f is onto, we must show that for any choice (z,w) in $\mathbb{R}^2$, we can find an (x,y) so that $f(x,y) = (z,w)$. So, choose (z,w), and it is easy to see that

$$f(w, z - w^2 + 1) = (w^2 - 1 + (z - w^2 + 1), w) = (z,w)$$

as desired.

4. *(16 points)* In the Deep Blue Chess club, there are 7 boys and 3 girls. In how many different ways can a team of 4 players be chosen if there must be at least one girl on the team? Explain your answer.

Solution: The simplest way to count the teams with at least one girl is to count all teams, and subtract off the number of teams with no girls. Since there are 10 possible players, the number of teams of 4 that can be formed is $\binom{10}{4}$, and the number of teams with all boys is $\binom{7}{4}$. Hence, the number of teams with at least one girl is

$$\binom{10}{4} - \binom{7}{4} = 210 - 35 = 175$$

Alternatively, we could add together the number of teams with 1 girl and 3 boys, with 2 girls and 2 boys, and with 3 girls and 1 boy.

$$\binom{3}{1}\binom{7}{3} + \binom{3}{2}\binom{7}{2} + \binom{3}{3}\binom{7}{1}$$

I'll leave it to you to check that this is 175 possibilities.

A common error (I made it myself, at first) is to reason as follows: Pick the girl (there are 3 choices here), then pick the rest of the team without regard to gender (there are $\binom{9}{3}$ ways to do this), giving a total of $3\binom{9}{3}$ possible teams. However, this overcounts teams with more than one girl. To see that, notice that the team where you pick Ann first, and then pick Joe, Jim, and Eve is exactly the same team as the one where you picked Eve first, and then Joe, Jim, and Ann.

5. *(16 points)* Let S_n be the n^{th} Fibonacci number, that is, an element of the sequence $1, 1, 2, 3, 5, 8, \ldots$ given by

$$S_n = \begin{cases} 1 & \text{if } n = 0 \text{ or } n = 1 \\ S_{n-1} + S_{n-2} & \text{if } n > 1 \end{cases}$$

Use induction to prove that for any $n \geq 0$, we have $S_{2n} \geq 2^n - 1$. (Note that this bound is on the even terms only.)

Solution: First, we show the base case, which is $n = 0$: $S_0 \geq 2^0 - 1$ since $S_0 = 1$ and $2^0 - 1 = 0$.

Next, we show the induction step. That is, we assume that $S_{2n} \geq 2^n - 1$, and establish that $S_{2(n+1)} \geq 2^{n+1} - 1$.

$$\begin{aligned}
S_{2(n+1)} &= S_{2n+2} \\
&= S_{2n+1} + S_{2n} && \text{by the definition of } S_{2n+2} \\
&= S_{2n} + S_{2n-1} + S_{2n} && \text{by the definition of } S_{2n+1} \\
&= 2S_{2n} + S_{2n-1} \\
&\geq 2(2^n - 1) + S_{2n-1} && \text{applying the inductive hypothesis} \\
&= 2^{n+1} - 2 + S_{2n-1} \\
&\geq 2^{n+1} - 1 && \text{since } S_{2n-1} \geq 1 \text{ when } n \geq 1
\end{aligned}$$

6. *(16 points)* Consider the relation given by $a \circ b \Leftrightarrow 3a + 5b$ is even. Is this an equivalence relation? Prove your answer.

Solution: Yes, it is. One way to do this is to see that $a \circ b$ holds exactly when a and b have the same parity (that is, they are both even or both odd). To show that, first suppose a and b are both even. Then $5a$ and $3b$ are both even, and so is their sum. In the case where a and b are both odd, $5a$ and $3b$ are also odd, and the sum of two odd numbers is even, so $a \circ b$ in this case, too. However, if a and b have different parity (one even, one odd), then so will $5a$ and $3b$, and their sum will be odd.

Given that, checking that we have an equivalence relation is easy: It is reflexive, since a is always the same parity as itself, it is symmetric, since if a and b are the same parity, so are b and a. Transitivity is almost as easy: if a and b are the same parity, and if b and c are the same parity then a and c are, since they are both the same parity as b.

If you didn't see this, you can check each of the properties directly.

Refl: $(a \circ a)$ Note that $3a + 5a = 8a$, which is always even if a is an integer.

Symm: (If $a \circ b$, then $b \circ a$) Suppose $3a + 5b$ is even. Now notice that

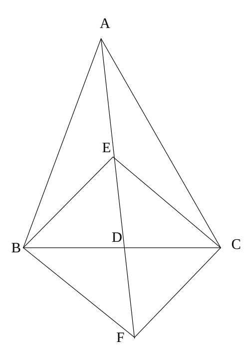
$$5a + 3b = (8a - 3a) + (8b - 5b) = 8(a + b) - (3a + 5b)$$

Since $3a + 5b$ is even by assumption, and $8(a + b)$ is even, then so is their sum. Hence $5a + 3b$ is even, as desired.

Trans: (If $a \circ b$ and $b \circ c$, then $a \circ c$). We assume $3a + 5b$ and $3b + 5c$ are both even. Thus, so is their sum, which is $3a + 5b + 3b + 5c = 3a + 8b + 5c$. Hence, since $8b$ is always even, $3a + 5c$ must be as well.

7. *(16 points)* Let AD be the median from vertex A of $\triangle ABC$. Let E be the midpoint of AD. Extend AD past D to F so that $|ED| = |DF|$. Draw the line segments BE, BF, FC and CE.

a. Draw the picture described above.



b. Prove that $m\angle BEF = m\angle DFC$.

Solution: Since AD is the median of $\triangle ABC$, $|BD| = |DC|$. Angles $\angle BDE$ and $\angle FDC$ are vertical angles, and hence have the same measure, and $|ED| = |DF|$ by construction. Applying SAS, we have $\triangle BDE \cong \triangle CDF$. Hence, $m\angle BEF = m\angle DFC$, because these are corresponding angles in congruent triangles.

Extra Credit. In this problem, the boxes have returned, but this time there are four of them! There is a statement in each of the four boxes, but three of the statements are false, and one is true. Exactly one of the boxes is worth 2 points; the others are worth no points. If you place an X in the box worth 2 points, you can have them.

Box A	Box B	Box C	Box D
Box B is worth 2 points	Box D is worth 2 points	This box is worth no points	The statement in Box A is false
		X	

For 8 additional points, you must give a proof that the box you picked was the proper one, and that there was only one choice.

Solution: First, notice that if Box D is true, then since there is only one true statement, Box C must contain a false statement, and hence is worth 2 points.

If Box D is false, then Box A contains a true statement. But then Box C must contain a false statement, and again is worth 2 points. This contradicts the fact that only one box is worth 2 points, since Box A says that Box B is worth 2 points.

Hence, Box D must contain the true statement, and Box C is worth 2 points.