2. Triangles and congruence of triangles

2.1. Basic measurements. Three distinct lines, a, b and c, no two of which are parallel, form a triangle. That is, they divide the plane into some number of regions; exactly one of them, the triangle, is bounded, and has segments of all three lines on its boundary.

The triangle with vertices A, B, C is denoted by $\triangle ABC$, where A is the point of intersection of the lines b and c; B is the point of intersection of the lines a and c; and C is the point of intersection of the lines a and b. These points of intersection divide each of the lines into two unbounded half-lines and one bounded line segment, called a side of the triangle.

The triangle, $\triangle ABC$, defines 6 numbers, the angle measures (also called the angles) at the vertices A, B and C, and the lengths of the sides, which are the line segments BC, AC and AB.

The angle measure at for example the vertex A is denoted by $m\angle A$, or $m\angle BAC$.

2.2. Historical note. The use of the phrase "measure of an angle" is relatively modern. Up to about 50 years ago, the measure of the angle at A was simply denoted by A or $\angle A$, and it was left to the reader to distinguish between the angle and its measure. When convenient, we will follow this convention, and use the same notation for an angle and its measure.

2.3. More on measurements. We will always give angle measures in radians, so, if A B and C all lie on a line, with B between A and C, then $m\angle ABC = \pi$.

We denote the length of the side AB, for example, by $|AB|$. Until modern times, the side and its length were denoted by the same symbol, and the reader had to figure out which is which from the context. As with angles, when convenient, we will also use the same notation for a line segment and its length.

The pair of lines, a and b, for example, determines two angles; the question of which of these angles is determined by the triangle can be stated in words with difficulty; we will leave this as visually obvious.

2.4. Congruence. Two triangles, $\triangle ABC$ and $\triangle A'B'C'$, are congruent if the corresponding angles have equal measures, and the corresponding sides have equal lengths. That is, the triangles, $\triangle ABC$ and $\triangle A'B'C'$ are congruent if $m\angle A = m\angle A'$; $m\angle B = m\angle B'$; $m\angle C = m\angle C'$; $|AB| = |A'B'|$; $|AC| = |A'C'|$; and $|BC| = |B'C'|$. In this case, we write $\triangle ABC \cong \triangle A'B'C'$.

For physical triangles, two triangles are congruent if they exactly match if you put one on top of the other. Another way of saying this, for ideal triangles, is that there is an isometry of the plane (a composition of rotation, translation and reflection) that maps one exactly onto the other.

Exercise 2.1: Show that congruence of triangles is an equivalence relation.

2.5. Important remark about notation. It is essentially obvious that congruence of triangles is an equivalence relation. However, the statement that $\triangle ABC \cong \triangle A'B'C'$ says nothing about whether $\triangle BCA$ is or is not congruent to $\triangle A'B'C'$. More precisely, the statement $\triangle ABC \cong \triangle A'B'C'$ not only tells you that these two triangles are congruent, but also tells you that $m\angle A = m\angle A'$, $|AB| = |A'B'|$, etc.

2.6. The axiom for congruence.

Axiom 5 (ASA)**.** *If* $m\angle A = m\angle A'$, $m\angle B = m\angle B'$ *and* $|AB| = |A'B'|$, *then* $\triangle ABC \cong \triangle A'B'C'$.

It is common to refer to the above angle as "Angle-Side-Angle" or ASA.

For physical triangles this is essentially obvious. If you know the length of a side, and you know the two angles, then the lines on which the other sides lie are determined, so the third vertex is also determined.

2.7. Exercises. A physical triangle is determined by 6 pieces of information, the 3 lengths and the 3 angles. There are 6 possible statements concerning 3 pieces of information. Convince yourself that AAA and SSA are false, while AAS, ASA, SAS and SSS are true. (There is nothing here for you to hand in, but you need this information for the next two questions.)
Remark: One of these, AAS, is not obvious; in fact it is false in spherical geometry.

In the following few exercises, when you are asked to prove something you may assume that AAS, ASA, SAS and SSS are true. One other fact that you may use is Thm. 3.4: the sum of the angles of a triangle is π. Note that *this is only for these exercises*; in general we cannot assume things we have not proven or taken as an axiom, because we may wind up applying circular reasoning (that is, giving a proof that something is true which implicitly assumes it was true to begin with.) But the main point of this exercise is to get you thinking about how geometry works, so we can relax our restrictions a little.

Exercise 2.2: Is it true that no 2 pieces of information suffice to determine a triangle? That is, can you find two pieces of information so that if you have any two triangles for which these two measurements are the same, the triangles must necessarily be congruent. Prove your answer.

Exercise 2.3: What about 4 pieces of information; i.e., do any four pieces of information suffice for congruence of triangles? Prove your answer.

A quadrilateral is a region bounded by four line segments; that is, it is a four-sided figure. The quadrilateral with vertices, A, B, C and D, in this order, is determined by the four line segments connecting A and B, B and C, C and D, and connecting D and A. For $ABCD$ to form a quadrilateral, these segments must not intersect except at the verticies.

A quadrilateral defines 8 pieces of information: the lengths of the four sides, and the measures of the four angles. Two quadrilaterals are congruent if these 8 pieces of information agree.

What is the minimal number of pieces of information one needs about two quadrilaterals to prove that they are congruent? (No response needed here, but you need the answer for the next question.)

Exercise 2.4: State and prove one congruence theorem for quadrilaterals, where the hypothesis consists of the minimal number of pieces of information.

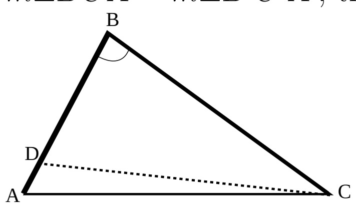
2.8. Monotonicity of lengths and angles. Here are two more axioms we shall need. Essentially, they say that for every real number, a segment can be scaled to that length, and that angles can be subdivided into angles of any measurement between 0 and π.

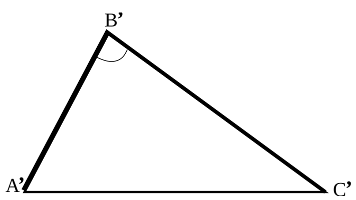
Axiom 6 (Ruler Axiom)**.** *If A B and C are distinct points on a line, in that order, then* $|AB| < |AC|$. *Further, for every positive real number* $r < |AC|$, *there is a point B between A and C so that* $|AB| = r$.

Axiom 7 (Protractor Axiom)**.** *If k is a line, and A is a point on k, then, for every number α with $0 < \alpha < \pi$, there is a line m through A so that the angles formed by k and m have measures α and $\pi - \alpha$. Further, if $0 < \beta < \alpha < \pi$, then there is a line n passing through the sector of angle α formed by k and m, so that n and k form an angle of measure β.*

Theorem 2.1 (SAS)**.** *If $\triangle ABC$ and $\triangle A'B'C'$ are such that $|AB| = |A'B'|$, $m\angle ABC = m\angle A'B'C'$, and $|BC| = |B'C'|$, then they are congruent.*

Proof. Suppose we are given two triangles $\triangle ABC$ and $\triangle A'B'C'$ as in the statement. If $m\angle BCA = m\angle B'C'A'$, then we would be done (by ASA).





So let us consider the case where they are different, and arrive at a contradiction. We may assume that $m\angle BCA > m\angle B'C'A'$ (if not, just exchange the names on the triangles).
Apply the second part of Axiom 7 to find a line passing through the point C and some point D lying between A and B, so that $m\angle BCD = m\angle B'C'A'$.
By ASA, $\triangle BCD \cong \triangle B'C'A'$. Therefore $|DB| = |A'B'|$. But we are given that $|A'B'| = |AB|$. Therefore, $|DB| = |AB|$. Since D lies on the line determined by A and B, and lies between them, this contradicts Axiom 6.

□

2.9. Isosceles triangles. A triangle is isosceles if two of its sides have equal length. The two sides of equal length are called legs; the point where the two legs meet is called the apex of the triangle; the other two angles are called the base angles of the triangle; and the third side is called the base.

While an isosceles triangle is defined to be one with two sides of equal length, the next theorem tells us that is equivalent to having two angles of equal measure.

Theorem 2.2 (Base angles equal)**.** *If $\triangle ABC$ is isosceles, with base BC, then $m\angle B = m\angle C$.*
Conversely, if $\triangle ABC$ has $m\angle B = m\angle C$, then it is isosceles, with base BC.

Exercise 2.5: Prove Theorem 2.2 by showing that $\triangle ABC$ is congruent to its reflection $\triangle ACB$. Note that there are two parts to the theorem, and so you need to give essentially two separate arguments.

2.10. Congruence via SSS.

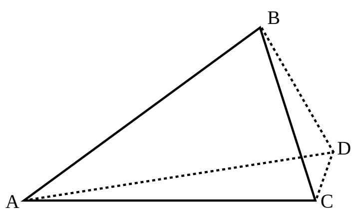
Theorem 2.3 (SSS)**.** *If $\triangle ABC$ and $\triangle A'B'C'$ are such that $|AB| = |A'B'|$, $|AC| = |A'C'|$ and $|BC| = |B'C'|$, then $\triangle ABC \cong \triangle A'B'C'$.*

Proof. If the two triangles were not congruent, then one of the angles of $\triangle ABC$ would have measure different from the measure of the corresponding angle of $\triangle A'B'C'$. If necessary, relabel the triangles so that $\angle A$ and $\angle A'$ are two corresponding angles which differ, with $m\angle A' < m\angle A$.

We find a point D and construct the line AD so that $m\angle DAB = m\angle A'$, and $|AD| = |A'C'|$. (That this can be done follows from Axioms 6 and 7.) It is unclear where the point D lies: it could lie inside triangle ABC; it could lie on the line BC between B and C; or it could lie on the other side of the line BC. We need to take up these three cases separately.

Exercise 2.6: Suppose the point D lies on the line BC. Explain why this yields an immediate contradiction.

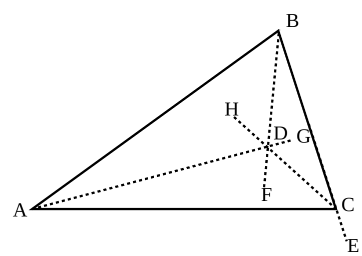
For both of the remaining cases, we draw the lines BD and CD. We observe that, by SAS, $\triangle ABD \cong \triangle A'B'C'$. It follows that $|BD| = |B'C'| = |BC|$ and that $|AD| = |A'C'| = |AC|$. Hence $\triangle BDC$ is isosceles, with base DC, and $\triangle ADC$ is isosceles with base CD. Since the base angles of an isosceles triangle have equal measure, $m\angle BDC = m\angle BCD$ and $m\angle ADC = m\angle ACD$.



First, we take up the case that D lies outside $\triangle ABC$; that is, D lies on the other side of the line BC from A.

Exercise 2.7: Finish this case of the proof, first by showing that $m\angle ADC < m\angle BDC$ and $m\angle BCD < m\angle ACD$. Then use the isosceles triangles to arrive at the contradiction that $m\angle ADC < m\angle ADC$.

We now consider the case where D lies inside $\triangle ABC$. Extend the line BC to some point E. Observe that $m\angle BCD + m\angle DCA + m\angle ACE = \pi$, from which it follows that $m\angle BCD + m\angle DCA < \pi$. Next, extend the line BD past D to some point F. Also extend the line AD past the point D to some point G, and extend the line CD past the point D to some point H.



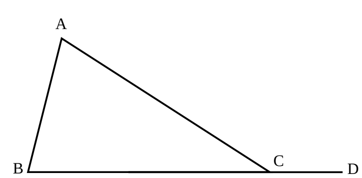
Exercise 2.8: Finish this case of the proof by explaining why $\pi < m\angle BDC + m\angle CDA$ and $m\angle BCD + m\angle DCA < \pi$, and then show that this leads to the contradiction $\pi < \pi$.

□

2.11. Inequalities for general triangles.

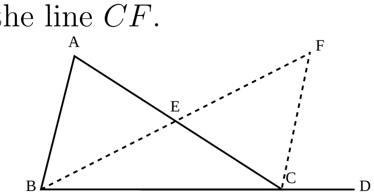
Theorem 2.4 (exterior angle inequality)**.** *Consider the triangle $\triangle ABC$. Let D be some point on the line BC, where C lies between B and D. Then*

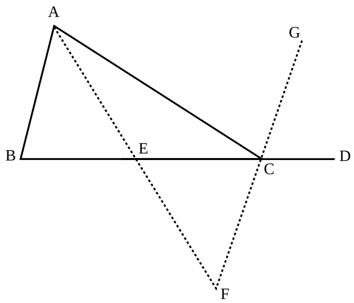
(i) $m\angle ACD > m\angle A$.
(ii) $m\angle ACD > m\angle B$.



Proof. We first prove part (i). Let E be the midpoint of the line segment AC; that is, E lies on the line AC, between A and C, and $|AE| = |EC|$. Draw the line BE and extend it past E to the point F, so that E is the midpoint of BF. Also draw the line CF.

Exercise 2.9: Finish the proof of part (i). Hint: First show that $\triangle AEB \cong \triangle CEF$ (Thm. 1.1 may be useful.) Use that to compare $m\angle A$ and $m\angle ECF$, and conclude that $m\angle ACD > m\angle ACF = m\angle A$.





For part (ii), we choose E to be the midpoint of the line BC, and extend AE beyond E to F, so that $|AE| = |EF|$. Also, extend the line Now extend the line FC beyond C to some point G.

Exercise 2.10: Finish the proof of part (ii). First show that $\triangle AEB \cong \triangle FEC$, and then compare $m\angle FCE$, $m\angle DCG$, $m\angle DCA$, and $m\angle B$.

□

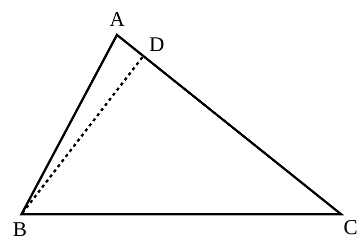
The next theorem says that in a triangle, if one angle is bigger than another, the side opposite the bigger angle must be longer than the one opposite the smaller angle. This is generalizes the fact that the base angles of isosceles triangles are equal (Thm. 2.2).

Theorem 2.5. *In $\triangle ABC$, if $m\angle A > m\angle B$, then we must have $|BC| > |AC|$.*

Proof. Assume not. Then either $|BC| = |AC|$ or $|BC| < |AC|$.

Exercise 2.11: Show that if $|BC| = |AC|$, the assumption $m\angle A > m\angle B$ is contradicted.

Exercise 2.12: Now assume $|BC| < |AC|$, find the point D on AC so that $|BC| = |CD|$, and draw the line BD. Finish the proof in this case. Hint: Use Thm. 2.4 and the fact that $|BD| = |CD|$ to conclude that $m\angle CDB > m\angle A$. Now observe that $m\angle DBC < m\angle ABC$. Explain why this gives the contradiction $m\angle CBD < \angle CBD$.



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The converse of the previous theorem is also true: opposite a long side, there must be a big angle.

Theorem 2.6. *In $\triangle ABC$, if $|BC| > |AC|$, then $m\angle A > m\angle B$.*

Proof. Assume not. If $m\angle A = m\angle B$, then $\triangle ABC$ is isosceles, with apex at C, so $|BC| = |AC|$, which contradicts our assumption.

If $m\angle A < m\angle B$, then, by the previous theorem, $|BC| < |AC|$, which again contradicts our assumption. □

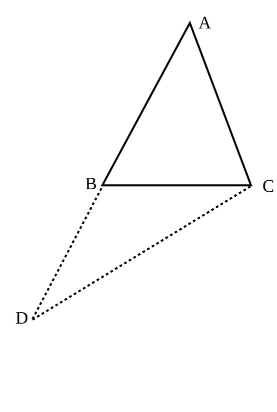
The following theorem doesn't quite say that a straight line is the shortest distance between two points, but it says something along these lines. This result is used throughout much of mathematics, and is referred to as "the triangle inequality".

Theorem 2.7 (the triangle inequality)**.** *In $\triangle ABC$, we have*

$$|AB| + |BC| > |AC|$$

.

Exercise 2.13: Prove the triangle inequality: First extend AB to a point D so that $|BD| = |BC|$, then form the isosceles triangle $\triangle BDC$. Use this triangle and Thm 2.2 to show that $m\angle ADC < m\angle ACD$. Conclude that $|AD| > |AC|$ by using another theorem from this section. Then show that $|AB| + |BC| > |AC|$.

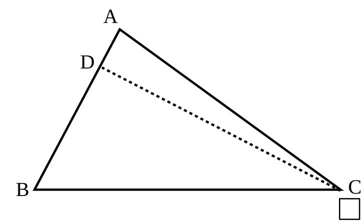


2.12. Congruence via AAS.

Theorem 2.8 (AAS). *Suppose we are given triangles ABC and $A'B'C'$, where $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, and $|BC| = |B'C'|$. Then $\triangle ABC \cong \triangle A'B'C'$.*

Proof. We first observe that, by either SAS or ASA, if $|AB| = |A'B'|$, then $\triangle ABC \cong \triangle A'B'C'$. Hence we can assume that $|AB| \neq |A'B'|$, from which it follows that either $|AB| < |A'B'|$ or $|AB| > |A'B'|$. We can assume without loss of generality that $|AB| > |A'B'|$ (that is, if we had that $|AB| < |A'B'|$, then we would interchange the labelling of the two triangles).

Now find the point D between A and B, so that $|BD| = |A'B'|$. Observe that, by SAS, $\triangle DBC \cong \triangle A'B'C'$. Hence $m\angle BDC = m\angle A' = m\angle A$. This contradicts that fact that, since $\angle BDC$ is an exterior angle for $\triangle ADC$, we must have that $m\angle BDC > m\angle A$. □



This concludes the generalities concerning congruence of triangles. We now know the four congruence theorems, ASA, SAS, SSS and AAS. We also know that the other two possibilities, SSA and AAA, are not valid. It follows that, for example, if we are given the lengths of all three sides of a triangle, then the measures of all three angles are determined. However, we do not as yet have any means of computing the measures of these angles in terms of the lengths of the sides.

2.13. Perpendicularity and orthogonality. Two lines intersecting at a point A are perpendicular or orthogonal if all four angles at A are equal. In this case, each of the angles has measure $\pi/2$. These angles are called right angles. It is standard in mathematics to use the words perpendicular and orthogonal interchangeably.

Basic Construction. Given a line k, and any point A, there is a line through A perpendicular to k.

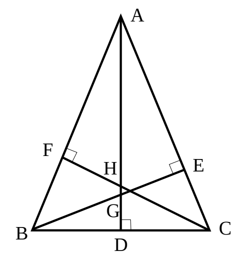
Exercise 2.14: Prove that the line through A perpendicular to k is unique. (Note that A may or may not lie on k.)

In any triangle, there are three special lines from each vertex. In $\triangle ABC$, the altitude from A is perpendicular to BC; the median from A bisects BC (that is, it crosses BC at a point D so that $|BD| = |DC|$); and the angle bisector bisects $\angle A$ (that is, if E is the point where the angle bisector meets BC, then $m\angle BAE = m\angle EAC$).

Theorem 2.9. *If A is the apex of the isosceles triangle ABC, and AD is the altitude, then AD is also the median, and is also the angle bisector, from A.*

Exercise 2.15: Prove this theorem. (Hint: Construct the altitude and apply AAS to the pair of resulting triangles.)

Theorem 2.10. *In an isosceles triangle, the three altitudes meet at a point.*



Proof. Let A be the apex of the isosceles $\triangle ABC$, and let AD be the altitude, which is also the median and the angle bisector. Similarly, let E be the endpoint on AC of the altitude from B, and let F be the endpoint on AB of the altitude from C. Let G be the point of intersection of AD with BE, and let H be the point of intersection of AD with CF. We need to prove that $G = H$.

By AAS, $\triangle FAC \cong \triangle EAB$. Hence $|AF| = |AE|$. Since AD is also the angle bisector, by ASA, $\triangle AFH \cong \triangle AEG$. Hence $|AH| = |AG|$, from which it follows that $G = H$. □

Exercise 2.16: Prove that the three angle bisectors in an isosceles triangle meet at a point.

Exercise 2.17: Prove that the three medians in an isosceles triangle meet at a point.