4. Lengths, areas and proportions

We now turn to a brief discussion of area. Rather than carefully developing this theory, we shall begin with some "obvious" facts, such as

(i) The area of a rectangle which has side lengths a and b is ab.
(ii) Congruent figures have equal area
(iii) The area of a region obtained as a union of two non-overlapping regions is the sum of the areas of these regions.

4.1. Right triangles.

A *right triangle* is one which contains a right angle. The two sides emanating from the right angle are called the *legs* of the triangle; the third side, opposite the right angle is the *hypoteneuse*.

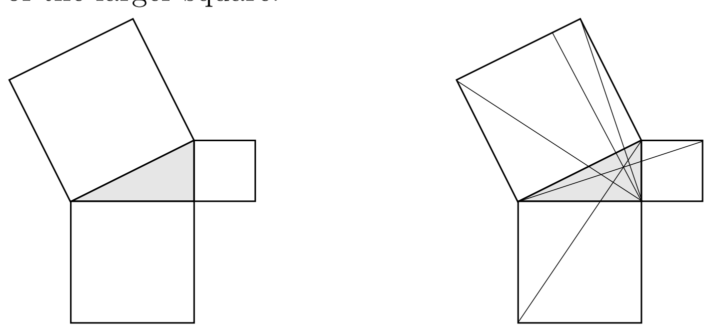
Proposition 4.1. *If the legs of a right triangle have lengths a and b, then the area of the triangle is $\frac{1}{2}ab$.*

Exercise 4.1: Prove this proposition by constructing an appropriate rectangle.

Proposition 4.2. *In $\triangle ABC$, if D is the endpoint of the altitude from A, then the area of $\triangle ABC$ is equal to $\frac{1}{2}|AD||BC|$.*

Exercise 4.2: Prove this proposition.

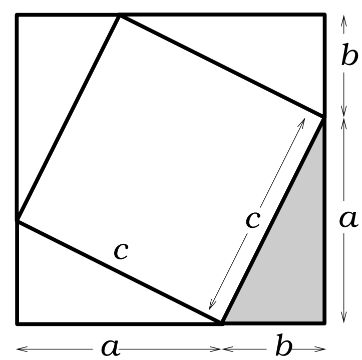
The Pythagorean Theorem, relating the lengths of the legs of a right triangle to the length of the hypotenuse, is very well-known; you've probably seen it repeatedly since your elementary school days, and know it quite well. The figure on the left below is sometimes described as a proof of the Pythagorean Theorem, but in fact it is much closer to a geometric statement of the theorem. To turn it into a proof requires constructing a number of additional lines, and a fairly complex argument. On left is the figure for Euclid's proof, sometimes called "the bride's chair". We won't give the argument here, but you might try to figure it out on your own. The idea is to show that there are two pairs of congruent triangles, and the area of each triangle in a pair is half of one of the smaller squares. The sum of the areas in the pairs gives the area of the larger square.



Theorem 4.3 (The Pythagorean Theorem). *Let the lengths of the legs of a right triangle be a and b, and let the length of the hypoteneuse be c. Then $a^2 + b^2 = c^2$.*

There are many proofs of this theorem; we know of at least 40. Below we give one of the simpler ones. It is necessary to use the Parallel Axiom (Axiom 8) axiom, either implicitly or explicitly, in order to prove the Pythagorean Theorem; the theorem is false in both spherical and hyperbolic geometry, which have a different version of Axiom 8. Surprisingly, the Pythagorean Theorem is equivalent to this axiom; that is, the Parallel Axiom can be proven if we assume the Pythagorean Theorem first.

Proof. The proof is essentially the figure shown. We start with a right triangle with legs of length a and b and hypotenuse of length c. Now we construct three more copies of this triangle, arranging them to construct a square of side length $a+b$ as in the figure. This is indeed a square because all four sides are of length $a+b$, and each angle of the outer quadrilateral is a right angle.



In addition, there is an inner quadrilateral formed whose sides are the hypotenuse of each of the triangles. This quadrilateral is certainly a rhombus, because each side is of length c. But it is in fact a square, because each of its angles are right: at each vertex there are three angles which sum to an angle of measure π. Two of these are the acute angles in a right triangle, and so the third must be of measure $\pi/2$.
The area of the outer square is $(a+b)^2$, the area of each of the four triangles is $\frac{1}{2}ab$, and the area of the inner square is c^2. Thus, we have

$$(a+b)^2 = 4(\frac{1}{2}ab) + c^2$$

From this, we readily see that $a^2 + b^2 = c^2$. $\square$

Exercise 4.3: Locate a different proof of the Pythagorean Theorem, and explain it in your own words.

4.2. Similar triangles. We say that two triangles ABC and $A'B'C'$ are similar if $m\angle A = m\angle A'$, $m\angle B = m\angle B'$ and $m\angle C = m\angle C'$.

Since the sum of the measures of a triangle is constant, if we are given two triangles, $\triangle ABC$ and $\triangle A'B'C'$, and we know that $m\angle A = m\angle A'$, and $m\angle B = m\angle B'$, then the triangles are similar.

If the triangles are similar, we write $\triangle ABC \sim \triangle A'B'C'$.

Proposition 4.4. *Let D be some point on the side AB of $\triangle ABC$, and let k be the line through D parallel to BC. Let E be the point where k crosses AC. Then $\triangle ABC \sim \triangle ADE$.*

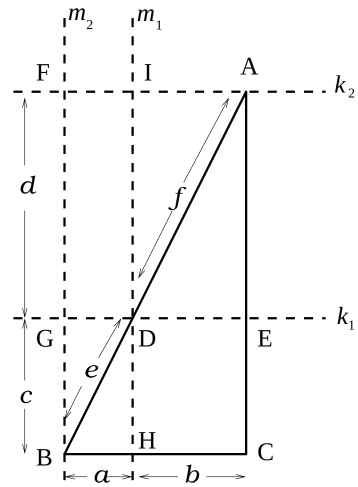
Exercise 4.4: Prove this proposition.

We want to prove the very useful fact that the side lengths of similar triangles are proportional. However, this is easier to do if we first prove it for right triangles, and then apply this result to the general case.

Lemma 4.5 (Ratios for right triangles)**.** *Let $\triangle ABC \sim \triangle A'B'C'$, where C and C' are right angles. Then*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}.$$

Proof. Without loss of generality, we can assume that $|AB| > |A'B'|$, and so we start with $\triangle ABC$, and construct another triangle which is congruent to $\triangle A'B'C'$ inside it.



Let k_1 be a line parallel to BC that meets AB at the point D where $|AD| = |A'B'|$. Denote the point where k_1 meets AC by E. Let k_2 be the line through A parallel to BC, and let m_2 be the line perpendicular to AC at B. Then, since AC and m_2 are both perpendicular to BC, they are parallel. Let F be the point of intersection of m_2 with k_2. Then $AFBC$ is a rectangle. Also, let G be the point of intersection of m_2 and k_1. Finally, let m_1 be the line perpendicular to k_1 and k_2 through D; denote the point of intersection of m_1 with BC by H and denote the point of intersection of m_1 with k_2 by I.
Since the lines BC, k_1 and k_2 are all parallel, and the lines AC, m_1 and m_2 are all parallel, $|BH| = |GD| = |FI|$, $|HC| = |DE| = |IA|$, $|BG| = |HD| = |CE|$, and $|GF| = |DI| = |EA|$. We give names to these quantities; we set $a = |BH| = |GD| = |FI|$, $b = |HC| = |DE| = |IA|$, $c = |BG| = |HD| = |CE|$ and $d = |GF| = |IE| = |EA|$. We also set $e = |BD|$ and $f = |DA|$.

The line AB divides the rectangle $AFBC$ into two congruent triangles. Since the areas of $\triangle BGD$ and $\triangle BHD$ are equal, and the areas of $\triangle IDA$ and $\triangle EAD$ are equal, we obtain that the areas of the two smaller rectangles are equal; that is, $ad = bc$.

Since we drew the line EG parallel to BC, we know that $\triangle AED \sim \triangle ACB$. So, for this case, our theorem, which we want to prove, says that

$$\frac{d}{c+d} = \frac{b}{a+b} = \frac{f}{e+f}.$$

To see that the first of these inequalities is true, notice that it holds when $d(a+b) = b(c+d)$ (by cross multiplying). But this holds since we have already established that $ad = bc$.

We check that the second equality is true by using the first equality together with the Pythagorean theorem. That is, we write $f^2 = b^2 + d^2$ and $(e+f)^2 = (a+b)^2 + (c+d)^2$, and then, as above, cross-multiply and use the two facts that we have already proven; namely that $ad = bc$, and that

$$\frac{d^2}{(c+d)^2} = \frac{b^2}{(a+b)^2}.$$

Now since $\triangle ABC \sim \triangle A'B'C'$, we have $m\angle A = m\angle A'$ and $m\angle B = m\angle B'$. Since k_1 is parallel to BC, $m\angle B = m\angle EDA$ (using alternate interior angles and vertical angles). Since $|AD| = |A'B'|$ by construction, $\triangle A'B'C' \cong \triangle ADE$ via ASA. Using the above, we conclude that

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}$$

as desired. □

Now that we have shown the desired property for right triangles, we can use that result to show it for arbitrary similar triangles.

Theorem 4.6 (Ratios for triangles). *Let $\triangle ABC \sim \triangle A'B'C'$. Then*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}.$$

Proof. As above, we can assume without loss of generality that $|AB| > |A'B'|$. Find the point D on AB so that $|AD| = |A'B'|$, and draw the line DE parallel to BC. As above, using alternate interior angles, we see that $m\angle ADE = m\angle ABC$ and $m\angle AED = m\angle ACB$. Hence $\triangle ADE \cong \triangle A'B'C'$, and so it suffices to show that

$$\frac{|AB|}{|AD|} = \frac{|BC|}{|DE|} = \frac{|CA|}{|EA|}.$$

Let $k = AF$ be the altitude from A, and let G be the point of intersection of k with the line DE. Then ADF and ADG are similar right triangles, and ACF and AEG are similar right triangles. Hence the proportion theorem for right triangles yields

$$\frac{|AD|}{|AB|} = \frac{|AG|}{|AF|} = \frac{|AE|}{|AC|}.$$

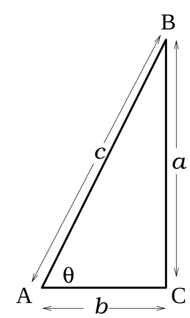
That $\frac{|DE|}{|BC|} = \frac{|AG|}{|AF|}$ follows easily from the Lemma above. □

Exercise 4.5: Prove that the lines joining the midpoints of the sides of a triangle divide the triangle into four congruent triangles.

4.3. BASIC TRIGONOMETRIC FUNCTIONS. Let $\triangle ABC$ be such that $\angle C$ is a right angle. Let $a = |BC|$, $b = |AC|$ and $c = |AB|$. Also, let $\theta = m\angle A$. Then, by the proportion theorem for right triangles, the ratios a/b, a/c, and b/c all depend only on the measure of the angle θ.

In what follows, we will deliberately confuse an angle with its measure; i.e., we will not distinguish between $\angle A$ and $m\angle A$. It should always be clear from the context which is meant.

For $0 < \theta < \pi/2$, we define the trigonometric functions as ratios of lengths in right triangles:



$\sin\theta = a/c$
$\cos\theta = b/c$
$\tan\theta = a/b = \sin\theta/\cos\theta$
$\cot\theta = b/a = 1/\tan\theta = \cos\theta/\sin\theta$
$\sec\theta = c/b = 1/\cos\theta$
$\csc\theta = c/a = 1/\sin\theta$

We note that it is almost immediate from the definition that

$$\sin(\pi/2 - \theta) = \cos\theta \qquad \text{and} \qquad \cos(\pi/2 - \theta) = \sin\theta.$$

In order to extend the definitions to all values of θ, first we set

$$\sin 0 = 0 \qquad \text{and} \qquad \sin\frac{\pi}{2} = 1$$

which is reasonable since the sine is small for small angles, and close to 1 for angles close to right angles. Now to define obtuse angles,

$$\sin(\pi - \alpha) = \sin(\alpha) \qquad \text{for} \qquad 0 \le \alpha \le \frac{\pi}{2}$$

and to get negative values, we let

$$\sin(-\theta) = -\sin(\theta).$$

Finally, since an angle of 2π represents a full turn, we say that $\sin(2\pi + \alpha) = \sin \alpha$. Putting these together defines the sine for all values.

We can then use the fact that $\sin(\pi/2 - \theta) = \cos\theta$ and $\tan\theta = \sin\theta/\cos\theta$ to extend the definitions of the other functions to all real numbers (except those where we would need to divide by 0, such as for $\tan\frac{\pi}{2}$.

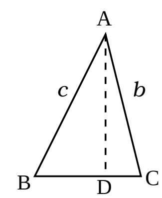
Exercise 4.6: Prove that, for any number θ,

$$\sin^2\theta + \cos^2\theta = 1.$$

Theorem 4.7 (Law of Sines)**.** *Let ABC be a triangle, with sides of length $a = |BC|$, $b = |AC|$ and $c = |AB|$. Then*

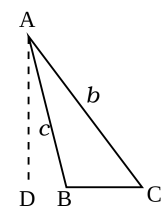
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Proof. We first assume that C is a right angle, and, that $c = 1$. Then $\sin A = a$ and $\sin B = b$, so the result holds in this case. Using the theorem of proportions for right triangles, our result holds for all right triangles.



Now let ABC be an arbitrary triangle, where B and C are acute angles (i.e., their measures are less than $\pi/2$), and let AD be the altitude from A. Then $\sin B = \frac{|AD|}{c}$, and $\sin C = \frac{|AD|}{b}$. Hence

$$\frac{\sin B}{b} = \frac{|AD|}{bc} = \frac{\sin C}{c}.$$



We next take up the case that angle B is an obtuse angle. We again drop the altitude AD from A to the line BC, but now this altitude lies outside $\triangle ABC$. Note that B lies between D and C. As above, using our known facts about right triangles, we obtain, $|AD| = |AC|\sin C = |AB|\sin B$, from which the desired result follows.

If angle C is obtuse, repeat the above argument, exchanging B and C.

The other equality follows by looking at the altitude from B and/or C. $\square$

Theorem 4.8 (Law of Cosines)**.** *Let ABC be a triangle, where none of the angles are right angles, with sides of length $a = |BC|$, $b = |AC|$ and $c = |AB|$. Then*

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Proof. As above, we need to worry about the possibility that $\angle C$ is obtuse. We first take up the case that it is acute. It could be that either $\angle A$ or $\angle B$ is obtuse, we note that our hypotheses, and the formula we wish to prove, remain unchanged if we interchange A and B; we assume that $\angle B$ is not obtuse. It then follows that if AD is the altitute from A, then D lies between A and B. Let $d = |AD|$. Let $a_1 = |BD|$, and let $a_2 = |DC|$. Then, by the Pythagorean theorem, $c^2 = a_1^2 + d^2$. Again by the Pythagorean theorem, $d = b^2 - a_2^2$. Substituting, we obtain

$$c^2 = a_1^2 + d^2 = a_1^2 + b^2 - a_2^2 = (a_1 + a_2)^2 + b^2 - 2a_1a_2 - 2a_2^2 = a^2 + b^2 - 2aa_2 = a^2 + b^2 - 2ab\cos C,$$

where, in the last equality, we have used the fact that $\cos C = a_2/b$.

We next take up the case that $\angle C$ is obtuse. In this case, the endpoint of the altitude AD is such that C lies between B and D. Similar to the above, we set $a_1 = |CD|$. Then the

Pythorean theorem yields the following.

$$(a + a_1)^2 + d^2 = c^2, \quad d^2 + a_1^2 = b^2.$$

Substituting, we obtain,

$$c^2 = (a + a_1)^2 + b^2 - a_1^2 = a^2 + b^2 + 2aa_1.$$

We note that $\cos(\pi - C) = a_1/b$. Hence, upon setting $\cos C = -\cos(\pi - C)$, we obtain,

$$c^2 = a^2 + b^2 + -2ab\cos(\pi - C) = a^2 + b^2 - 2ab\cos C.$$

□

4.4. OTHER IMPORTANT TRIGONOMETRIC FORMULAE. We will not prove the addition formulae for sines and cosines; however, since they are important, we will state them.

Theorem 4.9.

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha.$$

Theorem 4.10.

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

One can easily prove these theorems by constructing $\triangle ABC$, where $m\angle A = \alpha + \beta$, and the altitude from A divides this triangle into two triangles, $\triangle ABD$ and $\triangle ADC$, in such a way that $m\angle BAD = \alpha$ and $m\angle DAC = \beta$. Then, using the definitions of the sine and cosine for right triangles, the addition formula for the sine follows from the law of sines, and the addition formula for the cosine follows from the Pythagorean theorem and the law of cosines.

We can use the laws of sines and cosines to obtain information about triangles.

Exercise 4.7: This exercise tells you how to compute the other three pieces of information about a triangle if you are given SAS.

(a) Assume we are given the lengths a and b of triangle ABC, and we are given $\angle C$ (that is, we are given $\sin C$ and/or $\cos C$). Explain in words how we would find the length of the other side, and the other two angles. (Hint: Use the above theorems.)

(b) Find explicit formulas for these three pieces of information in explicit terms; that is, find a formula for c, and find formulas for the other two angles, in terms of a, b and either $\sin C$ or $\cos C$.

Exercise 4.8: Repeat the above for the congruence relation ASA. That is, if you are given the sines and or cosines of two of the angles, and given the length of the included side, find the sine or cosine of the other angle and find the lengths of the other two sides.

Exercise 4.9: Repeat the above exercise for AAS.

Exercise 4.10: Repeat the above exercise for SSS.