6. Some Amusements

6.1. Amusement 1: Fill in the steps in the construction below, and observe the result.

(1) Start with arbitrary triangle $\triangle T_1 = \triangle ABC$.
(2) Construct the lines k, m and n, parallel to AB, BC and CA, respectively.
(3) These three lines form a new triangle, $\triangle T_0 = \triangle A'B'C'$; label these so that $B'C'$ is parallel to BC, $A'C'$ is parallel to AC and $A'B'$ is parallel to AB.
(4) Observe that the sides of $\triangle T_1$ divide $\triangle T_0$ into four triangles.
(5) Since the sides are parallel, these four triangles are all similar to the big triangle; in particular, $\triangle A'B'C' \sim \triangle ABC$.
(6) Since they have some sides in common, the four smaller triangles are all congruent.
(7) It follows that A is the midpoint of $B'C'$; B is the midpoint of $A'C'$; and C is the midpoint of $A'B'$.
(8) Construct the perpendicular bisectors of $A'B'$, $B'C'$ and $A'C'$; we know these all meet at a point ; call it O. (O is the center of the circumscribed circle for $\triangle A'B'C'$; this center is called the *orthocenter*).
(9) The lines OA, OB and OC, when extended, are altitudes of $\triangle ABC$.
(10) We have shown that the three altitudes of an arbitrary triangle meet at a point.

6.2. Amusement 2:

(1) Let $\triangle ABC$ be an arbitrary triangle.
(2) Let AD and BE be medians. Let G be the point of intersection of these two lines.
(3) Draw the line DE.
(4) Observe that DE is parallel to AB. (This was part of a homework assignment.)
(5) Then $\triangle GAB \sim \triangle GDE$.
(6) We know that $|DE| = 1/2|AB|$. Hence, $|GD| = 1/2|AG|$ and $|GE| = 1/2|GB|$.
(7) Repeat the above argument, using the medians from A and C.
(8) Conclude that the three medians of an arbitrary triangle meet at a point. (This point is called the *centroid* of the triangle; it is at the center of gravity.)
(9) We also have shown that the centroid divides each median into two segments; the segment between the centroid and the vertex is twice as long as the segment between the centroid and the opposite side.

6.3. Circles and circles. Two circles Σ and Σ' either are disjoint, or they meet at a point, in which case they are said to be *tangent*, or they meet at two points, in which case, they *intersect*.

It is essentially immediate that two circles with the same center but different radii are disjoint.

Since a line is the shortest distance between two points, if we have two circles where the distance between the centers is greater than the sum of their radii, then the circles are necessarily disjoint.

If we have two circles with the property that the distance between their centers is exactly equal to the sum of their radii, then the line between their centers contains a point on both circles.

Proposition 6.1. *If two circles have three points in common, then they are identical.*

Proof. Label the three points as A, B and C, and draw the lines AB, BC and CA.

The three points cannot be collinear, for a line intersects a circle in at most two points. Since the three points are not collinear they form a triangle. Then both circles are circumscribed about $\triangle ABC$. Since the circumscribed circle about a triangle is unique, the two circles are the same. □

Proposition 6.2. *Suppose the circles Σ and Σ' intersect at the points A and B. Let O be the center of Σ and let O' be the center of Σ'. Then the line OO' is the perpendicular bisector of the line segment AB.*

Proof. Since AB is a chord of Σ (Σ'), the perpendicular bisector of the chord AB passes through the center O (O'). Hence the perpendicular bisector of the line segment AB is the line determined by O and O'. □

Proposition 6.3. *Suppose the circles Σ and Σ' are tangent at A. Then the line connecting the centers of these circles, passes through A.*

Proof. Let O be the center of Σ and let O' be the center of Σ'. Suppose the line OO' does not pass through A. Construct the perpendicular from A to the line OO', and let B be the point where this perpendicular bisector meets OO'. Now construct the point C on the line AB so that B lies between A and C and so that $|AB| = |BC|$. Then, by *sas*, $\triangle OAB \cong \triangle OCB$. Hence $|OC| = |OA|$, from which it follows that C lies on Σ. Using the same argument, $\triangle O'AB \cong \triangle O'CB$, from which it follows that C lies on Σ'. We have constructed a second point of intersection of Σ and Σ'; since we assumed these circles had only one point in common, we have reached a contradiction. □

Corollary 6.4. *If the circles Σ and Σ' are tangent at A, and k is the line tangent to Σ at A, then k is tangent to Σ'.*

Proof. Since Σ and Σ' are tangent at A the line OO' connecting their centers passes through A. Hence the radius of Σ at A lies on the line OO', and so OO' is orthogonal to k. The same argument shows that the radius of Σ' lies on the line OO'. Since the tangent to Σ' at A is the line orthogonal to the radius at A, it is k. □

We now return to the case of intersecting circles. Suppose the circles Σ and Σ' intersect at the points A and B. Then the line joining the centers O and O' is the perpendicular bisector of the line segment AB. We draw the radii, OA, OB, $O'A$ and $O'B$. We define the *angle of intersection* of these two circles at A to be $\pi - m\angle OAO'$. Likewise, the angle of intersection at B is $\pi - \angle OBO'$.

Remark: We could have chosen the angle between the circles to be $\angle OAO'$. The reason for our choice is that, if two circles are tangent, and each lies outside the other, then, by continuity, the angle between them is 0, while if one lies inside the other, then the angle between them is π.

Proposition 6.5. *If the circles Σ and Σ' intersect at A and B, then the angle of intersection at A has the same measure as the angle of intersection at B.*

Proof. We draw the line AB, which is a chord for both circles. We know that OO' is the perpendicular bisector of this chord; let C be the point of intersection of the lines AB and OO'.

Observe first that, by *sss*, $\triangle OAC \cong \triangle OBC$ and $\triangle O'AC \cong \triangle O'BC$. It follows that $m\angle OAC = m\angle OBC$ and that $m\angle O'AC = m\angle O'BC$. Hence $m\angle OAO' = m\angle OBO'$. □

Since the angle of intersection at A and the angle of intersection at B have the same measure, we can simply call it the *angle of intersection* of the two circles.

We remark that, since the tangent to Σ at A is orthogonal to OA, and the tangent to Σ' at A is orthogonal to $O'A$, then the angle between the lines OA and $O'A$ has the same measure as one of the angles between these tangents.

Proposition 6.6. *Suppose we are given three positive real number, a, b and c, where $a < c$, $b < c$ and $a + b > c$. Then there is a triangle with sides a, b and c.*

Proof. Consider the line segment AB, where $|AB| = c$. Draw the circle of radius a centered at B, and draw the circle of radius B centered at a. Since $a + b > c$ there are points on AB that lie inside both circles. Hence either one circle lies inside the other, or the circles intersect.

Since $a < c$, the point A lies outside the circle centered at B, and since $b < c$, the point B lies outside the circle centered at A. Hence neither circle lies inside the other, and so the circles intersect. Let C be one of the points of intersection, and observe that $|AC| = b$ and $|BC| = a$. □

6.4. Orthogonal circles. Two circles are *orthogonal* if the angle between them is $\pi/2$.

Proposition 6.7. *Let Σ be a circle with center O and radius r. Let A be some point on Σ, and let $r' > 0$ be any real number. Then there is a unique circle Σ' of radius r', orthogonal to Σ, where the center O' of Σ' lies on the line OA, and O' lies on the same side of O as does A.*

Proof. We first prove uniqueness. Suppose we have such a circle Σ'. Let A be one of the two points of intersection of Σ and Σ'. Then the triangle $OO'A$ is a right triangle with right angle at A. Hence, by the Pythagorean theorem, $|OO'|^2 = r^2 + (r')^2$. This shows that the distance from O to O' is determined; hence the circle Σ' is determined.

To prove existence, find the point O' on OA, on the same side of O as A, and at distance

$$\sqrt{r^2 + (r')^2}$$

from O. Then construct the circle Σ' of radius r' at that point.

To show that Σ and Σ' intersect, it suffices to show that there is a point B on both Σ and Σ', or equivalently, that there is a triangle with side lengths, r, r' and $|OO'|$. This follows from the above proposition, once we observe that $r < \sqrt{r^2 + (r')^2}$, $r' < \sqrt{r^2 + (r')^2}$, and $\sqrt{r^2 + (r')^2} < r + r'$. □

We remark without proof that, given two orthogonal circles Σ and Σ', there is a 1-parameter family of circles orthogonal to both Σ and Σ'. However, given three mutually orthogonal circles, there is no fourth circle orthogonal to all three.

6.5. Tangent circles.

Proposition 6.8. *Let Σ be a given circle of radius r and center O. Let A be any point, where $A \neq O$ and A does not lie on Σ. Then there is a circle Σ', centered at A, where Σ' and Σ are tangent.*

Proof. Draw the line OA. This line intersects Σ in two points; let B be one of them. Draw the circle Σ' of radius $|AB|$ about A. This circle certainly meets Σ at B. Since B lies on the line connecting the centers of the circles, the circles are tangent at B. □

We remark that we have in fact shown that there are exactly two circles centered at A that are tangent to Σ.

There are three possible orientations for the two tangent circle. We can have that Σ lies outside Σ' and Σ' lies outside Σ, or we can have that Σ lies inside Σ', or we can have that Σ' lies inside Σ.

Exercise: Suppose A, B and C are three given points on a line. How many distinct triples of mutually tangent circles are there, where one of the circles is centered at A, one is centered at B, and the third is centered at C.

Proposition 6.9. *Let $\triangle ABC$ be given. Then there are three mutually tangent circles, Σ_A centered at A, Σ_B centered at B, and Σ_C centered at C. If we require that each of the three circles lies outside the others, then the radii of these circles are determined by the lengths of the sides of the triangle.*

Proof. We need to find the radii; call these α, β and γ, where α is the radius of Σ_A, β is the radius of Σ_B and γ is the radius of Σ_C. Then we must solve the equations:

$$\alpha + \beta = |AB|, \quad \beta + \gamma = |BC|, \quad \gamma + \alpha = |CA|.$$

It is an exercise in linear algebra to show that these equations have a unique solution. $\square$

We close with the remark that, given three mutually tangent circles, there exist exactly two disjoint circles that are tangent to all three. If one has four mutually tangent circles, then there can be no fifth.