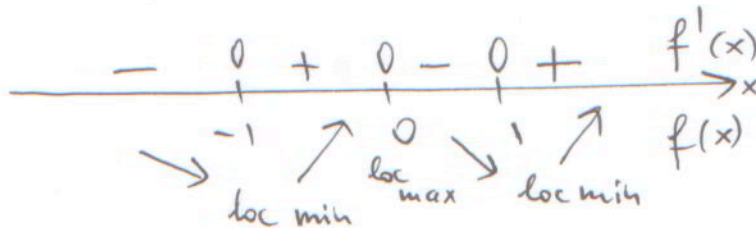Name _____

Problem 1 (10pt). Let $f(x) = x^4 - 2x^2$.

a) Find all the extreme points of $f(x)$ and determine their type (local maximum or local minimum)

$f'(x) = 4x^3 - 4x = 4x(x-1)(x+1)$

critical points: $x = 0$, $x = \pm 1$

$$\begin{array}{ccccccc} - & 0 & + & 0 & - & 0 & + \quad f'(x) \\ & -1 & & 0 & & 1 & \longrightarrow x \\ \searrow & \nearrow & & \searrow & & \nearrow & f(x) \\ \text{loc min} & & \text{loc max} & & \text{loc min} & & \end{array}$$

$\boxed{\begin{array}{l} x = 0 \quad \text{is a loc max} \\ x = \pm 1 \quad \text{are loc min} \end{array}}$
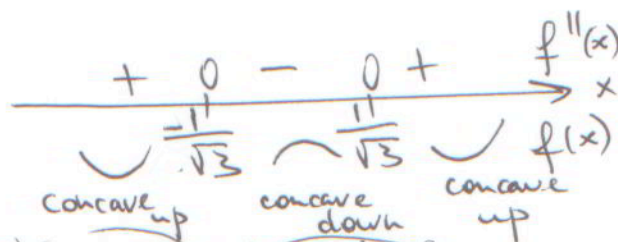
b) Where is $f(x)$ increasing or decreasing?

on $[-1, 0] \cup [1, \infty)$

on $(-\infty, -1] \cup [0, 1]$

c) Find all the inflection points.
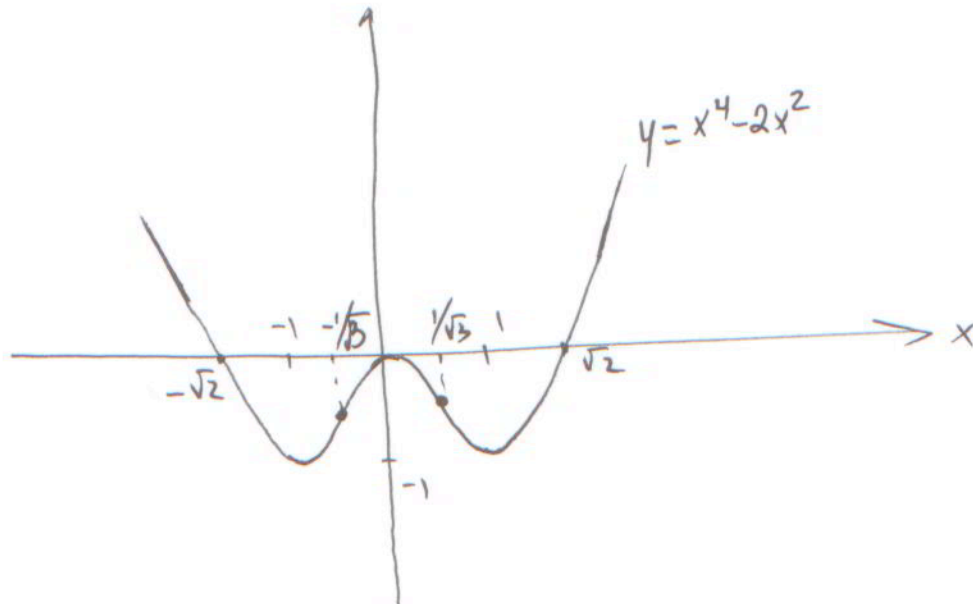
$f''(x) = 12x^2 - 4 = 4(3x^2 - 1)$

$$\begin{array}{ccccc} + & 0 & - & 0 & + \quad f''(x) \\ & \frac{-1}{\sqrt{3}} & & \frac{1}{\sqrt{3}} & \longrightarrow x \\ \smile & & \frown & & \smile \quad f(x) \\ \text{concave up} & & \text{concave down} & & \text{concave up} \end{array}$$

$\boxed{\begin{array}{l} x = \pm \frac{1}{\sqrt{3}} \quad \text{are} \\ \text{inflection pts} \end{array}}$

d) Where is $f(x)$ concave up or concave down?

on $(-\infty, -\frac{1}{\sqrt{3}}] \cup [\frac{1}{\sqrt{3}}, \infty)$

on $[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$

Name _____

e) Use the information above to draw the graph of $f(x)$.

$y = x^4 - 2x^2$

$f(0) = 0$

$f(\pm 1) = 1 - 2 = -1$

f is even

$f(x) = 0 \iff x^4 - 2x^2 = 0$

$x^2(x^2 - 2) = 0$

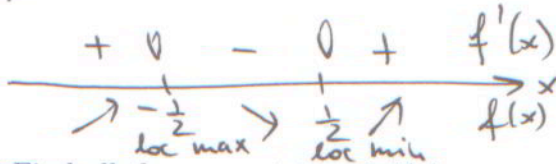zeros of f: $x = 0, \ x = \pm \sqrt{2}$

Name _____

Problem 2 (15pt). Let $f(x) = x - \arctan 2x$.

a) Find all the extreme points of $f(x)$.

$$f'(x) = 1 - \frac{2}{1+4x^2} = \frac{4x^2-1}{1+4x^2}$$

$$f'(x) = 0 \iff 4x^2 - 1 = 0 \iff x = \pm\frac{1}{2}$$

$x = -\frac{1}{2}$ is a loc max

$x = \frac{1}{2}$ is a loc min

b) Find all the asymptotes of $f(x)$.

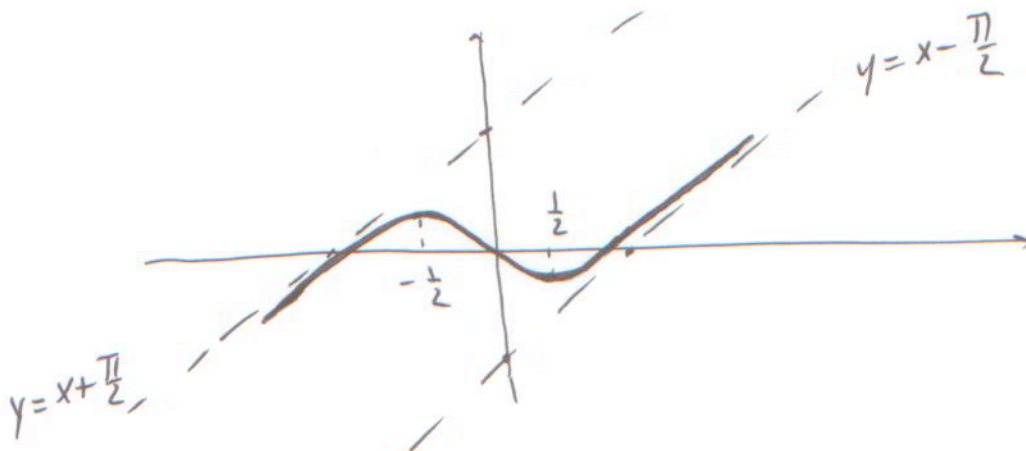No vertical asymptotes, no horizontal asymptotes.

Oblique asymptotes: $y = kx + b$

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \left(1 - \frac{\arctan 2x}{x}\right) = 1$$

$$b = \lim_{x \to \pm\infty} (f(x) - kx) = \lim_{x \to \pm\infty} (x - \arctan 2x - x) = -\lim_{x \to \pm\infty}(\arctan 2x) = \mp\frac{\pi}{2}$$

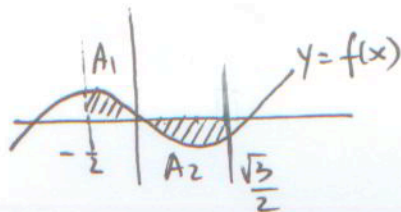$\boxed{y = x - \frac{\pi}{2}}$ is an oblique asymptote at $+\infty$, $\boxed{y = x + \frac{\pi}{2}}$ is an oblique as. at $-\infty$

c) Draw the graph of $f(x)$.

$y = x - \frac{\pi}{2}$

$\frac{1}{2}$

$-\frac{1}{2}$

$y = x + \frac{\pi}{2}$

d) Use your picture above to explain the geometrical meaning of the integral

$$\int_{-1/2}^{\sqrt{3}/2} (x - \arctan 2x)\, dx = A_1 - A_2$$

where A_1, A_2 are shadowed areas

$f\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2} - \arctan \sqrt{3} = \frac{\sqrt{3}}{2} - \frac{\pi}{3} < 0$

A_1

$y = f(x)$

$-\frac{1}{2}$

A_2

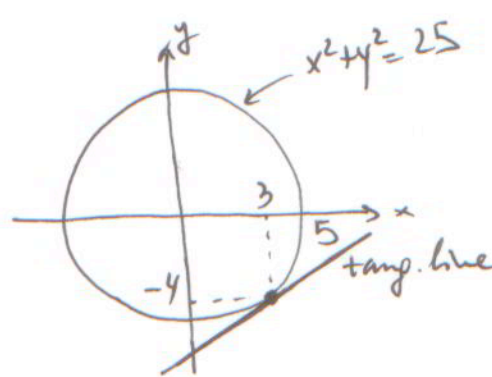$\frac{\sqrt{3}}{2}$

Name _____

Problem 3 (10pt). Let $x^2 + y^2 = 25$. Find the derivatives $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $(x, y) = (3, -4)$ and explain their geometrical meaning.

$\dfrac{d}{dx}$ ⟶ $x^2 + y^2 = 25$

$2x + 2y \dfrac{dy}{dx} = 0 \implies \boxed{\dfrac{dy}{dx} = -\dfrac{x}{y}}$

$\dfrac{dy}{dx}\Big|_{\substack{x=3 \\ y=-4}} = -\dfrac{3}{-4} = \boxed{\dfrac{3}{4}}$ is the slope of the tangent line to the circle $x^2 + y^2 = 25$ at pt $(3, -4)$

tang. line

OR $\dfrac{d}{dx}$

$\dfrac{d^2y}{dx^2} = -\dfrac{y - xy'}{y^2}$

$\dfrac{d^2y}{dx^2}\Big|_{\substack{x=3 \\ y=-4}} = -\dfrac{-4 - 3 \cdot \frac{3}{4}}{16} =$

$= \dfrac{25}{64}$

$\dfrac{d}{dx}$ ⟶ $2 + 2\dfrac{dy}{dx} \cdot \dfrac{dy}{dx} + 2y \dfrac{d^2y}{dx^2} = 0$

$1 + \left(\dfrac{dy}{dx}\right)^2 + y \dfrac{d^2y}{dx^2} = 0$

At $x = 3, y = -4,$

$1 + \underbrace{\left(\dfrac{dy}{dx}\Big|_{\substack{x=3 \\ y=-4}}\right)^2}_{\frac{9}{16}} + (-4) \cdot \dfrac{d^2y}{dx^2}\Big|_{\substack{x=3 \\ y=-4}} = 0$

$\dfrac{d^2y}{dx^2}\Big|_{\substack{x=3 \\ y=-4}} = \boxed{\dfrac{25}{64}} > 0 \implies$ the curve is concave up near pt $(3, -4)$

Name _____

Problem 4 (20pt). Calculate the following limits:

a) $\lim\limits_{x \to 0} \dfrac{e^x - \cos x}{\sin x}$ $= \left[\dfrac{1-1}{0} = \dfrac{0}{0}\right] = \lim\limits_{x \to 0} \dfrac{e^x + \sin x}{\cos x} = \dfrac{e^0 + 0}{1} = \boxed{1}$

b) $\lim\limits_{x \to \infty} \left(1 + \dfrac{2}{x}\right)^x = \lim\limits_{x \to \infty} y$, where

$y = \left(1 + \dfrac{2}{x}\right)^x$

$\ln y = x \ln\left(1 + \dfrac{2}{x}\right) = \dfrac{\ln\left(1 + \dfrac{2}{x}\right)}{\frac{1}{x}}$

$\lim\limits_{x \to \infty} \dfrac{\ln\left(1 + \dfrac{2}{x}\right)}{\frac{1}{x}} = \left[\dfrac{0}{0}\right] = \lim\limits_{x \to \infty} \dfrac{\dfrac{-\frac{2}{x^2}}{1 + \frac{2}{x}}}{-\frac{1}{x^2}} = \lim\limits_{x \to \infty} \dfrac{2}{1 + \frac{2}{x}} = 2$

$y = e^{\ln y} \xrightarrow{x \to \infty} 2 = \boxed{e^2}$

c) $\lim\limits_{x \to 0} \dfrac{\sin 2x}{\arcsin 4x} = \left[\dfrac{0}{0}\right] =$

$\lim\limits_{x \to 0} \dfrac{2\cos 2x}{4\sqrt{1 - 16x^2}} = \dfrac{2}{4} = \boxed{\dfrac{1}{2}}$

d) $\lim\limits_{x \to \infty} (\sqrt{x+1} - \sqrt{x}) = \lim\limits_{x \to \infty} \dfrac{(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}{\sqrt{x+1} + \sqrt{x}} = \lim\limits_{x \to \infty} \dfrac{1}{\sqrt{x+1} + \sqrt{x}} = \boxed{0}$

Name _____

Problem 5 (20pt). Find the largest possible area of a rectangle inscribed in the ellipse $\frac{x^2}{4} + y^2 = 1$.
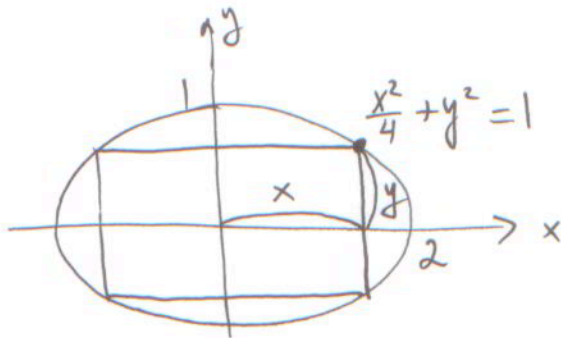
$$\frac{x^2}{4} + y^2 = 1 \implies y = \sqrt{1 - \frac{x^2}{4}} = \frac{1}{2}\sqrt{4 - x^2}$$

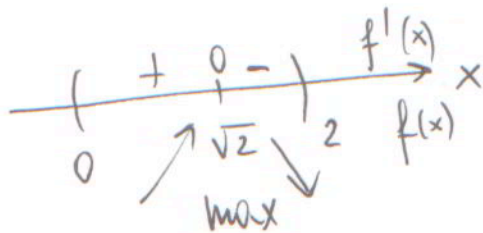$$\text{Area} = 2x \cdot 2y = 2x\sqrt{4 - x^2} = 2\sqrt{4x^2 - x^4}$$

$$A(x) = 2\sqrt{4x^2 - x^4}$$

Surch for $\max\limits_{0 < x < 2} f(x)$, where

$$f(x) = 4x^2 - x^4$$

$$f'(x) = 8x - 4x^3 = 4x(2 - x^2)$$

$$f(\sqrt{2}) = 4 \cdot 2 - 4 = 4$$

$\max\limits_{(0,2)} f(x) = 4$ at $x = \sqrt{2} \implies$

$\max\limits_{(0,2)} A(x) = 2 \cdot \sqrt{4} = 4$ at $x = \sqrt{2}$.

Answer: the largest area is $\boxed{4}$.

Name _____

Problem 6 (15pt). Evaluate the following integrals:

a) $\displaystyle\int_1^e \frac{\ln^2 x}{x}\,dx \quad = \quad \begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \\ x=1 \Rightarrow u=0 \\ x=e \Rightarrow u=1 \end{bmatrix} \quad = \quad \int_0^1 u^2\,du = \left.\frac{u^3}{3}\right|_0^1 = \boxed{\frac{1}{3}}$

b) $\displaystyle\int \cos^2 3x\,dx \quad = \quad \int \frac{1+\cos 6x}{2}\,dx \quad = \quad \frac{1}{2}x + \frac{1}{12}\sin 6x + C$

c) $\displaystyle\int_{-1}^1 \frac{x^3}{\sqrt{x^2+1}}\,dx \quad = 0 \qquad \text{since} \qquad \frac{x^3}{\sqrt{x^2+1}} \text{ is odd and}$

$[-1,1]$ is symmetric

Name _____

Problem 7 (10pt). Express the integral $\int_{-1}^{4} (4-x^2)\,dx$ as a limit of a Riemann sum. Show on a picture the lower Riemann sum and the upper Riemann sum for $n = 5$.
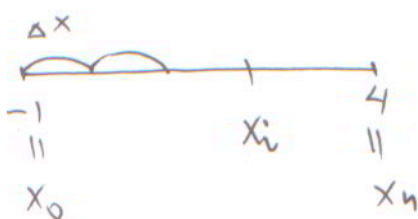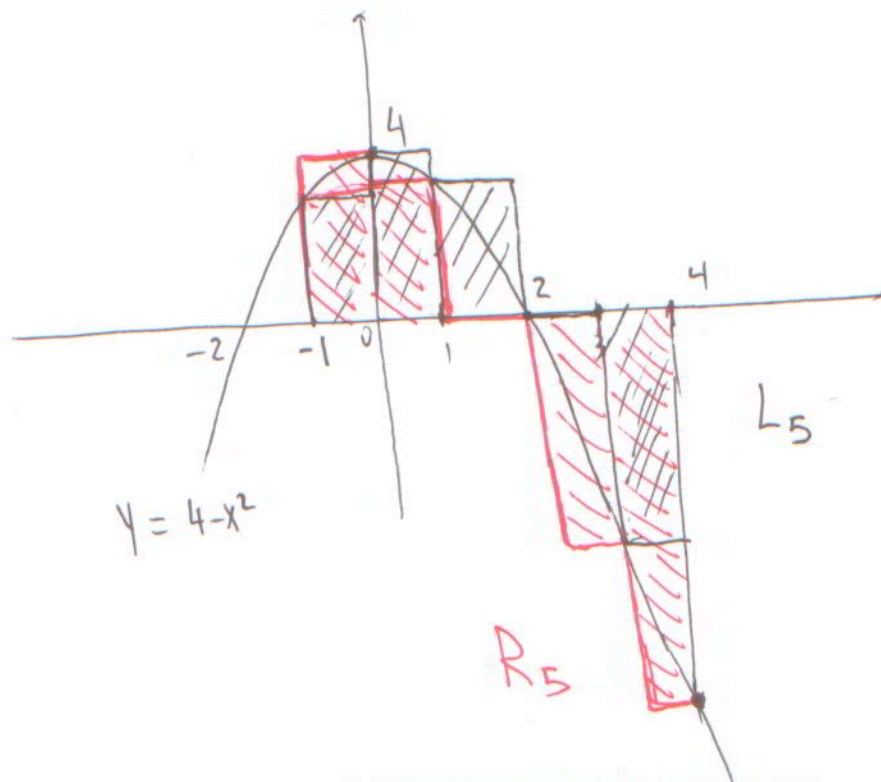
$$\Delta x = \frac{4-(-1)}{n} = \frac{5}{n}$$

$$x_i = x_0 + i\Delta x = -1 + i\frac{5}{n}$$

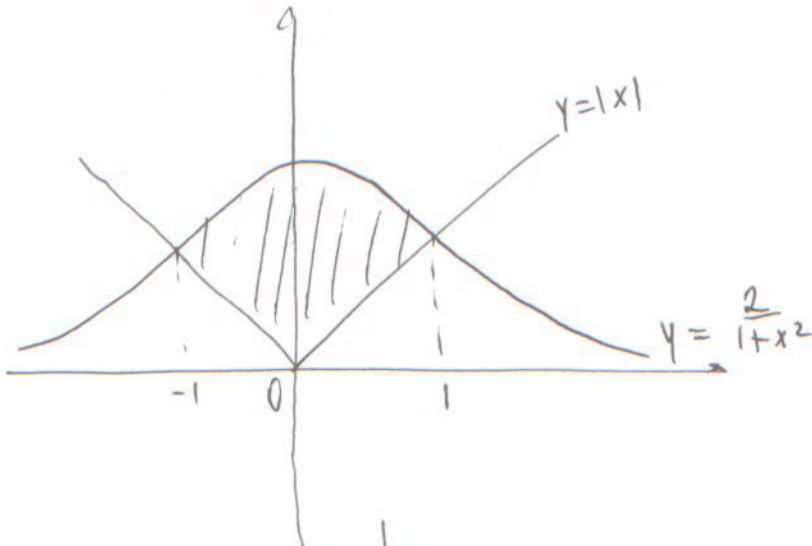$$\int_{a}^{b} f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$\int_{-1}^{4} (4-x^2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(4 - \left(-1 + \frac{5i}{n}\right)^2\right)\cdot \frac{5}{n}$$

$y = 4 - x^2$

Name _____

Problem 8 (This is an extra problem, 15pt. No partial credit.)

Find the area of the region bounded by the curves $y = \dfrac{2}{1+x^2}$ and $y = |x|$.

$$\text{Area} = 2 \int_0^1 \frac{2}{1+x^2} - x \, dx = 2 \left[2 \arctan x - \frac{x^2}{2} \right]_0^1 =$$

$$= 2 \left[2 \cdot \frac{\pi}{4} - \frac{1}{2} \right] = \boxed{\pi - 1}$$