MATH 132 Solutions to Midterm 2 (papaya)

10 pts 1. According to the poem by Ogden Nash,

Big fleas have little fleas,
Upon their backs to bite 'em,
And little fleas have lesser fleas,
And so, ad infinitum.

Assume each flea has exactly two fleas which bite it. If the largest flea weighs 0.8 grams, and each flea is 1/8 the weight of the flea it bites, what is the total weight of all the fleas?

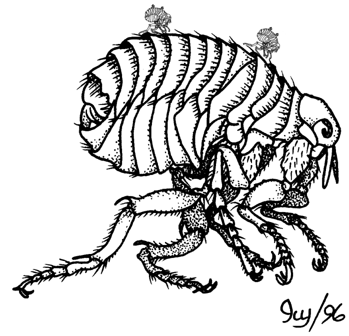


image adapted from
http://biodidac.bio.uottawa.ca

Solution: This is a geometric series, with a ratio of $2/8$ and constant term 0.8. Specifically, the mass of all the fleas is

$$0.8 + 0.8 \cdot \frac{1}{4} + 0.8 \cdot \frac{1}{4^2} + \ldots = 0.8 \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{0.8}{1 - 1/4} = \frac{0.8}{3/4} = \frac{32}{30} = \frac{16}{15}.$$

10 pts 2. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges to $\frac{\pi^2}{6}$. How many terms are necessary so that $\sum_{n=1}^{K} \frac{1}{n^2}$ is within $1/1000$ of $\pi^2/6$?

Solution: Here we use the integral to estimate the remainder of the series. We know that

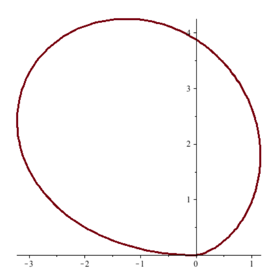
$$\sum_{n=K+1}^{\infty} \frac{1}{n^2} < \int_K^{\infty} \frac{dx}{x^2} = -\frac{1}{x}\bigg|_K^{\infty} = \frac{1}{K}.$$

So, we need K so that $1/K \leq 1/1000$, in other words, we need to sum to $n = 1000$.

10 pts 3. Calculate the area lying inside the part of polar curve $r = \theta^2(\pi - \theta)$ where $r > 0$ and $\theta > 0$.

Solution: We have $r \geq 0$ for $\theta < \pi$, so the area will be

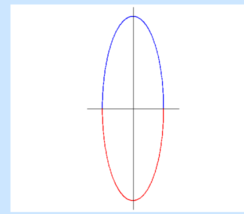
$$\begin{aligned}
\int_0^{\pi} \left[\theta^2(\pi - \theta)\right]^2 \frac{d\theta}{2} &= \int_0^{\pi} \theta^4(\pi - \theta)^2 \frac{d\theta}{2} \\
&= \frac{1}{2} \int_0^{\pi} \pi^2\theta^4 - 2\pi\theta^5 + \theta^6 \, d\theta \\
&= \frac{\pi^2\theta^5}{10} - \frac{\pi\theta^6}{6} + \frac{\theta^7}{14}\bigg|_0^{\pi} \\
&= \frac{\pi^7}{10} - \frac{\pi^7}{6} + \frac{\pi^7}{14} \\
&= \frac{\pi^7}{210}
\end{aligned}$$



10 pts

4. The set of points for which $y^2 = 1 - 4x^2$ is an ellipse. Write an integral which represents the circumference of the ellipse (that is, the length of the curve around its boundary). **You do not have to evaluate this integral.**

Solution: The function $f(x) = \sqrt{1-4x^2}$ describes the top half of the ellipse, and $g(x) = -\sqrt{1-4x^2}$ is the bottom half. Because of symmetry, we can compute just one half and double our answer.



Note that the domain of this function is $-1/2 \le x \le 1/2$, and $f'(x) = \dfrac{-4x}{\sqrt{1-4x^2}}$. Thus, the arc length is given by

$$2\int_{-1/2}^{1/2} \sqrt{1+[f'(x)]^2}\,dx = 2\int_{-1/2}^{1/2} \sqrt{1+\frac{16x^2}{1-4x^2}}\,dx$$

10 pts

5. Determine the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-7)^n}{n}$

Solution: As usual, we first use the ratio test to determine the radius of convergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2x-7)^{n+1}}{n+1}\cdot\frac{n}{(2x-7)^n}\right| = \lim_{n\to\infty}\left|\frac{n(2x-7)}{n+1}\right| = |2x-7|.$$

This ratio will be less than 1 when $-1 < 2x-7 < 1$, that is when $3 < x < 4$.

Now we need to establish what happens at the endpoints.

At $x = 4$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges (it is the harmonic series, or a p-series with $p = 1$).

At $x = 3$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, which converges by the alternating series test (the absolute values are decreasing, and $\lim 1/n = 0$).

This means the interval of convergence is $[3, 4)$, that is $3 \le x < 4$.

15 pts

6. For each of the series below, determine whether it converges or diverges. You must fully justify your answer to get any credit (that is, indicate what test you used, etc.).

(a) $\displaystyle\sum_{n=2}^{\infty}\frac{2}{n\ln n}$

Solution: This is easiest by the integral test:

$$\int_2^{\infty}\frac{2dx}{x\ln x} = 2\int_{\ln 2}^{\infty}\frac{du}{u} = 2(\lim_{M\to\infty}\ln|\ln M| - \ln|\ln 2|)$$

which diverges to $+\infty$. Thus, the original integral **diverges**.

(Strictly speaking, to use the integral test, the function must be decreasing. But since this is obvious for this function, since both n and $\ln n$ are increasing).

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+5}{(n^3+2)(n^3+3)}$

Solution: Here I find the limit comparison test easiest. If we ignore all the constants, we have

$$\frac{n^3+5}{(n^3+2)(n^3+3)} \sim \frac{n^3}{n^6} = \frac{1}{n^3}.$$

Now, we confirm that this is a valid series to compare to:

$$\lim_{n\to\infty} \frac{\frac{n^3+5}{(n^3+2)(n^3+3)}}{1/n^3} \lim_{n\to\infty} \frac{(n^3+5)n^3}{(n^3+2)(n^3+3)} = 1.$$

Since $\sum \frac{1}{n^3}$ is a convergent p-series (with $p = 3$), the original series **converges**.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^3}$

Solution: This one is a bit trickier: the terms of the series are not alternating, so we cannot use the alternating series test, nor are they always positive, so we cannot use the integral test or the comparison tests directly. Nor is the ratio test any help. Instead, let's look at the series of absolute values: $0 \leq \frac{|\cos n|}{n^3} \leq \frac{1}{n^3}$. Since $\sum \frac{1}{n^3}$ is a convergent p-series, we know that $\sum \frac{|\cos n|}{n^3}$ converges by the comparison test. But, since the series of absolute values converge, the original series is absolutely convergent, that is, it **converges**.

$\boxed{\text{10 pts}}$ 7. Calculate the volume of the solid obtained by revolving the area between the curves $y = 2x$ and $y = x^2$ around the line $x = -1$.

Solution: We determine where the two curves cross: $2x = x^2$ when $x = 0$ and $x = 2$. If we do this by "cylindrical shells", we have cylinders of height $2x - x^2$, and radius $x + 1$. This means that we obtain the volume

$$\int_0^2 2\pi(x+1)(2x-x^2)\,dx = 2\pi \int_0^2 2x + x^2 - x^3\,dx = 2\pi\ (x + \frac{x^3}{3} - \frac{x^4}{4})\Big|_0^2 = 2\pi(8/3) = \frac{16\pi}{3}$$

If instead we choose to do this by "washers", we have $0 \leq y \leq 4$. For a slice at height y, the inner radius is $x/2 + 1$, and the outer radius is $\sqrt{x} + 1$. This means the area will be given by

$$\int_0^4 \pi(\sqrt{y}+1)^2 - \pi(y/2+1)^2\,dy = \pi\int_0^4 y + 2\sqrt{y} + 1 - (y^2/4 + y + 1)\,dy = \pi\left(\frac{4}{3}y^{3/2} - \frac{y^3}{12}\right)\Big|_0^4 = \frac{16\pi}{3}$$