MATH 132 Solutions to Midterm 1 (electric)

20 pts 1. $\displaystyle\int \frac{3x-4}{x^3-x}\,dx$

Solution: We do this by partial fractions. Since $x^3 - x = x(x-1)(x+1)$, we want to find numbers A, B, and C so that

$$\frac{3x-4}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}.$$

Cross-multiplying, we have $A(x-1)(x+1) + Bx(x+1) + Cx(x-1) = 3x - 4$.

$$\begin{aligned} \text{If } x = 0, \text{ we have} \quad & -A = -4 \quad && \text{so } A = 4 \\ \text{If } x = 1, \text{ we have} \quad & 2B = -1 \quad && \text{so } B = -1/2 \\ \text{If } x = -1, \text{ we have} \quad & 2C = -7 \quad && \text{so } C = -7/2. \end{aligned}$$

This means that

$$\int \frac{3x-4}{x^3-x}\,dx = \int \frac{4}{x} - \frac{1/2}{x-1} - \frac{7/2}{x+1}\,dx = 4\ln|x| - \frac{1}{2}\ln|x-1| - \frac{7}{2}\ln|x+1| + C.$$

20 pts 2. $\displaystyle\int_3^6 \frac{dr}{(r-4)^4}$

Solution: Since $\dfrac{1}{(r-4)^4}$ becomes undefined when $r = 4$, we have write the integral as the sum of two improper integrals. That is,

$$\begin{aligned} \int_3^6 \frac{dr}{(r-4)^4} &= \int_3^4 \frac{dr}{(r-4)^4} + \int_4^6 \frac{dr}{(r-4)^4} \\ &= \lim_{t\to 4^-} \int_3^t \frac{dr}{(r-4)^4} + \lim_{t\to 4^+} \int_t^6 \frac{dr}{(r-4)^4} \\ &= \lim_{t\to 4^-} \left.\frac{-1}{3(r-4)^3}\right|_3^t + \lim_{t\to 4^+} \left.\frac{-1}{3(r-4)^3}\right|_t^6 \\ &= \lim_{t\to 4^-} \frac{-1}{3(t-4)^3} - \frac{1}{3} - \frac{1}{12} + \lim_{t\to 4^+} \frac{1}{3(t-4)^3}. \end{aligned}$$

However, both of the limits above are undefined, so the integral diverges.

20 pts 3. $\displaystyle\int_1^2 \frac{\sqrt{t^2-1}}{t}\,dt$

Solution: We make the substitution $t = \sec\theta$, and so $dt = \sec\theta\tan\theta\,d\theta$. Also, when $t = 1$ we have $\theta = 0$, and when $t = 2$ we have $\theta = \pi/3$. Thus, we have

$$\begin{aligned}\int_1^2 \frac{\sqrt{t^2-1}}{t}\,dt &= \int_0^{\pi/3} \frac{\sqrt{\sec^2\theta-1}}{\sec\theta}\sec\theta\tan\theta\,d\theta = \int_0^{\pi/3}\sqrt{\tan^2\theta}\tan\theta\,d\theta \\ &= \int_0^{\pi/3}\tan^2\theta\,d\theta = \int_0^{\pi/3}(\sec^2\theta-1)\,d\theta \\ &= \left.\tan\theta-\theta\right|_0^{\pi/3} = \tan(\pi/3)-\frac{\pi}{3} = \sqrt{3}-\frac{\pi}{3}.\end{aligned}$$

20 pts 4. $\displaystyle\int x^5\cos(x^3)\,dx$

Solution: First, we make the substitution $w = x^3$, so $dw = 3x^2\,dx$.

$$\begin{aligned}\int x^5\cos(x^3)\,dx &= \frac{1}{3}\int w\cos w\,dw = \frac{1}{3}\left(w\sin w - \int \sin w\,dw\right) \\ &= \frac{1}{3}(w\sin w + \cos w) + C = \frac{x^3\sin x^3 + \cos x^3}{3} + C\end{aligned}$$

where above we used integration by parts, with $u = w$ and $dv = \cos w\,dw$, so $du = dw$ and $v = \sin w$.

20 pts 5. $\displaystyle\int \cos^3(x)\sin^2(x)\,dx$

Solution: We use the identity $\cos^2 x + \sin^2 x = 1$ to transform all but one of the $\cos x$ terms into powers of $\sin x$. So we have

$$\int \cos^3(x)\sin^2(x)\,dx = \int (1-\sin^2 x)\sin^2 x\cos x\,dx.$$

Now we make the subsitution $u = \sin x$ with $du = \cos x\,dx$ to get

$$\int (1-u^2)u^2\,du = \int (u^2-u^4)\,du = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

20 pts 6. $\int_0^1 xe^{3x}\,dx$

Solution: We integrate by parts, with $u = x$ and $dv = e^{3x}\,dx$, so $du = dx$ and $v = \frac{1}{3}e^{3x}$. This gives us

$$\int_0^1 xe^{3x}\,dx = \left.\frac{x}{3}e^{3x}\right|_0^1 - \frac{1}{3}\int_0^1 e^{3x}\,dx = \left.\frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}\right|_0^1 = \frac{e^3}{3} - \frac{e^3}{9} + \frac{1}{9}$$

7. The values of a function $f(x)$ are given by the table at right.

x	-0.5	0	0.5	1	1.5	2	2.5
$f(x)$	-1	0	3	2	3	-1	1

15 pts (a) Use Simpson's rule to approximate $\int_0^2 f(x)\,dx$.

Solution: First, observe that there are two "extra" values in the table, namely $x = -0.5$ and $x = 2.5$. If we were to include those values (as some people did), the approximation would be for $\int_{-0.5}^{2.5} f(x)\,dx$.
We have $\Delta x = 1/2$, and $n = 4$. Simpson's rule is given by

$$S_4 = \frac{1/2}{3}\left(f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)\right) = \frac{1}{6}\left(0 + 12 + 4 + 12 - 1\right) = \frac{27}{6}.$$

5 pts (b) If you know that $-5x^2 \le f^{(4)}(x) \le 5x^2$, what is the maximum error in the approximation above?[1]

Solution: This is just a matter of filling in the proper values into the given formula. Since $|f^{(4)}(x)| \le 5x^2$, it takes on its maximum at $x = 2$, so we may use $K_4 = 5 \cdot 2^2 = 20$. This means the maximum error in the above approximation is

$$\frac{2^5 \cdot 20}{180(4)^4} = \frac{1}{72}.$$

20 pts 8. Does the improper integral $\int_2^\infty \frac{x^{1/2}+1}{5x-4}\,dx$ converge? Fully justify your answer (note that if the integral converges, you need not give its value).

Solution: No, **it diverges.** Note that $\frac{x^{1/2}+1}{5x^2-4} > \frac{x^{1/2}+1}{5x^2} > \frac{x^{1/2}}{5x^2} > \frac{1}{5x^{1/2}}$.
But $\frac{1}{5}\int_2^\infty \frac{dx}{x^{1/2}}$ diverges, since

$$\int_2^\infty \frac{dx}{x^{1/2}} = \lim_{M\to\infty} \left. 2x^{1/2}\right|_2^M = \lim_{M\to\infty} 2\sqrt{M} + 2\sqrt{2}$$

which diverges to $+\infty$ as $M \to \infty$. By the comparison test, the original integral diverges.

[1] Feel free to use the fact that $E_s < \frac{(b-a)^5}{180n^4}K_4$, where $K_4 = \max|f^{(4)}(x)|$ for $a \le x \le b$.

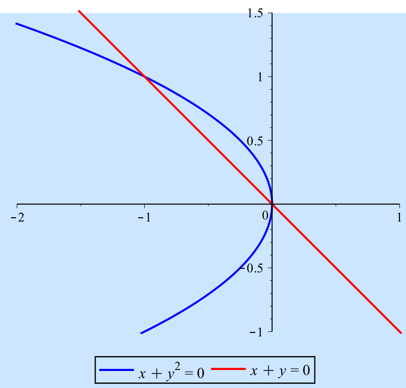
Alternatively, many people did the following instead:

$$\int_2^\infty \frac{x^{1/2}+1}{5x-4}\,dx = \int_2^\infty \frac{x^{1/2}}{5x-4}\,dx + \int_2^\infty \frac{1}{5x-4}\,dx$$
$$= \int_2^\infty \frac{x^{1/2}}{5x-4}\,dx + \lim_{M\to\infty} \frac{1}{5}\ln|5M-4| - \frac{\ln(6)}{5}$$

Since the integral in the last term is positive and the limit diverges to $+\infty$, the original integral must diverge. This method is, in fact, equivalent to comparison with $\int_2^\infty \frac{dx}{5x-4}$.

20 pts 9. Find the area lying between the two curves $x + y^2 = 0$ and $x + y = 0$

Solution: The curves in question are graphed at right. They intersect at $(0,0)$ and at $(-1,1)$: since the second curve is $y = -x$, substituting this into the first curve gives $x + x^2 = 0$, which holds when $x = 0$ or $x = -1$. We can choose to integrate either with respect to x or with respect to y.



Integrating with respect to x, we rewrite the curves as

$$y = \pm\sqrt{-x} \qquad y = -x.$$

The upper curve is $y = \sqrt{-x}$ and the lower curve is $y = -x$, and the relevant x-values are $-1 \le x \le 0$. The resulting integral is

$$\int_{-1}^0 \sqrt{-x} + x\,dx = -\frac{2}{3}(-x)^{3/2} + \frac{x^2}{2}\Big|_{-1}^0 = -\frac{2}{3} + \frac{1}{2} = \frac{1}{6}.$$

If we choose instead to integrate with respect to y, the rightmost curve is $x = -y^2$ and the leftmost curve is $x = -y$, and the relevant y-values are $0 \le y \le 1$. So the integral is

$$\int_0^1 -y^2 + y\,dy = -\frac{y^3}{3} + \frac{y^2}{2}\Big|_0^1 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}.$$