MAT 132 FINAL EXAM

NAME: SOLUTIONS SECTION: 2002

You have $2\frac{1}{2}$ hours to complete this exam. You may NOT use a calculator. You may NOT use any books or notes. Please SHOW YOUR WORK and EXPLAIN YOUR REASONING wherever possible. It might be helpful to use the following trigonometric identities:

$$\sin^2(x) + \cos^2(x) = 1$$
$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$
$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

THIS IS AN EASY EXAM, JUST A BIT LONG.

	1 15 pts	2 15 pts	3 15 pts	4 15 pts	5 20 pts	6 15 pts	7 10 pts	8 15 pts	9 15 pts
Score									

	10 10 pts	11 20 pts	12 20 pts	13 15 pts	14 30 pts	15 25 pts	16 25 pts	17 +20EC pts	**Total** 280 pts
Score									

NAME: SOLUTIONS SECTION:

1. (15 points) Evaluate $\int x^5 \cdot \ln(x)\,dx$.
(hint: use integration by parts)

$u = \ln x \qquad dv = x^5 dx$

$du = \frac{1}{x} dx \qquad v = \frac{1}{6}x^6$

SO $\int x^5 \ln x\,dx = \frac{1}{6}x^6 \ln x - \int \frac{1}{6}x^6 \left(\frac{1}{x}\right) dx$

$= \frac{1}{6}x^6 \ln x - \frac{1}{6}\int x^5 dx$

$\boxed{= \frac{1}{6}x^6 \ln x - \frac{1}{36}x^6 + C.}$

2. (15 points) Evaluate $\int_0^\pi \sin(2x)\,dx$.

$u = 2x$ IF $x = \pi$, $u = 2\pi$

$du = 2dx$ $x = 0$, $u = 0$

SO $\int_0^\pi \sin(2x)\,dx = \frac{1}{2}\int_0^{2\pi} \sin u\,du$

$= -\frac{1}{2}\cos u \Big|_0^{2\pi}$

$= -\frac{1}{2}(1-1) = \boxed{0.}$

3. (15 points) Evaluate $\int x^3\sqrt{3x^4+5}\,\mathrm{dx}$.

$$u = 3x^4 + 5$$
$$du = 12x^3\,dx$$

$$\int x^3\sqrt{3x^4+5}\,dx = \frac{1}{12}\int \sqrt{u}\,du$$
$$= \frac{1}{12}\left(\frac{2}{3}u^{3/2}\right) + C$$
$$= \boxed{\frac{1}{18}(3x^4+5)^{3/2} + C.}$$

4. (15 points) Evaluate the improper integral $\int_0^\infty 3e^{-x}\,\mathrm{dx}$.

$$\int_0^\infty 3e^{-x}\,dx = \lim_{M\to\infty}\int_0^M 3e^{-x}\,dx$$
$$= \lim_{M\to\infty}\left(-3e^{-x}\Big|_0^M\right.$$
$$= \lim_{M\to\infty}\left(-3e^{-M} + 3\right)$$
$$= 0 + 3 = \boxed{3}$$

 NAME: Sol. SECTION:

5. (20 points) Find a function $y(x)$ that satisfies the differential equation $y' = xy$ and the initial value $y(0) = 5$.

$$\frac{dy}{dx} = xy$$

$$\frac{dy}{y} = x\,dx$$

$$\int \frac{dy}{y} = \int x\,dx$$

$$\ln|y| = \tfrac{1}{2}x^2 + C$$

$$|y| = e^{\frac{1}{2}x^2 + C}$$

$$y = Ae^{\frac{1}{2}x^2}$$

A COULD BE + OR −

WHEN $x=0$, $y=5$, SO

$$5 = Ae^0$$

$$\Rightarrow A = 5.$$

THUS

$$\boxed{y(x) = 5e^{\frac{1}{2}x^2}}$$

CHECK: $y' = 5e^{\frac{1}{2}(x^2)}(x)$

$y' = y(x)$ ✓

6. (15 points) Last year I planted rhubarb in my garden and harvested 40 pounds of it. This year, I didn't plant any at all, but the rhubarb grew back anyway, and I harvested 30 pounds. I figure this pattern will continue; every year's harvest will be 75% of the previous year's harvest. If this pattern continues forever, what is the total yield (in pounds of rhubarb)?

$$\text{TOTAL} = \left(\text{LBS LAST YR}\right) + \left(\text{LBS THIS YR}\right) + \left(\text{LBS NEXT YR}\right) + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{\text{LBS}}{\text{YR}} = 40 + \tfrac{3}{4}40 + \left(\tfrac{3}{4}\right)^2 40 + \cdots$$

$$= 40\left(1 + \tfrac{3}{4} + \left(\tfrac{3}{4}\right)^2 + \cdots\right)$$

$$= 40\sum_{n=0}^{\infty} \left(\tfrac{3}{4}\right)^n$$

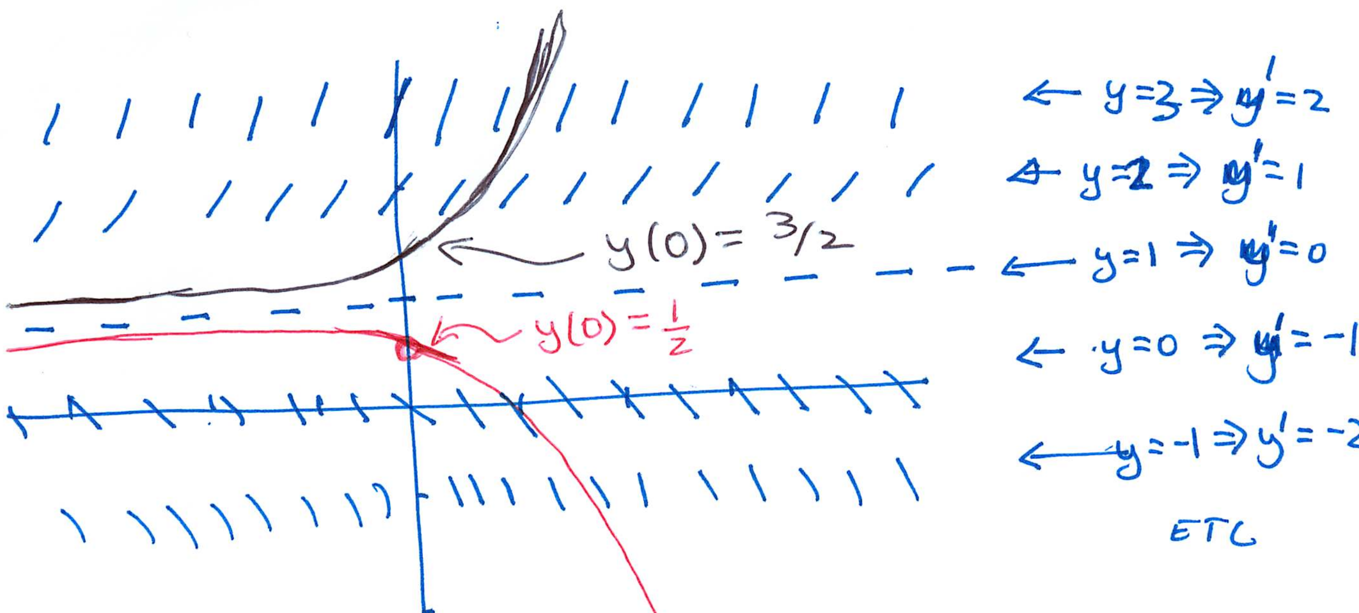
$$= \frac{40}{1 - 3/4} = \frac{40}{1/4} = \boxed{160 \text{ LBS}}$$

7. (10 points) Write an integral that equals the arclength of the graph of $y(x) = \ln(x)$ between $x = 1$ and $x = 3$. You do NOT need to solve this integral.

ARC LENGTH $= \int_a^b \sqrt{1+[f'(x)]^2}\, dx$

$$\int_1^3 \sqrt{1+\left(\frac{1}{x}\right)^2}\, dx$$

8. (15 points) Draw a slope-field for the differential equation $y' = y-1$. Use it to sketch two solution curves, one with $y(0) = 0.5$ and one with $y(0) = 1.5$



THIS IS EASY TO SOLVE DIRECTLY, SINCE IT IS SEPARABLE:

$y' = y-1$

$\frac{dy}{y-1} = dt$

$\ln|y-1| = t + C$

$y - 1 = Ae^t \quad \therefore \boxed{y = Ae^t + 1}$

SOLUTIONS ARE JUST SHIFTED EXPONENTIALS.

 NAME: SOL. SECTION:

9. (15 points) Does the series $\sum_{n=2}^{\infty} \frac{ln(n)}{n} = \frac{\ln(2)}{2} + \frac{\ln(3)}{3} + \frac{\ln(4)}{4} + \cdots$ converge or diverge? Explain why.

IT DIVERGES, SINCE FOR $n > 3$,

$\frac{\ln n}{n} > \frac{1}{n}$. SINCE THE HARMONIC SERIES $\sum \frac{1}{n}$ DIVERGES

$\sum_{n=2}^{\infty} \frac{\ln}{n}$ ALSO $\boxed{\text{DIVERGES}}$ (BY COMPARISON TEST)

10. (10 points) Does $\sum_{n=2}^{\infty}(-1)^n \frac{1}{\ln(n)} = \frac{1}{\ln(2)} - \frac{1}{\ln(3)} + \frac{1}{\ln(4)} - \cdots$ converge or diverge? Explain why.

THIS IS AN ALTERNATING SERIES.

WE HAVE $\frac{1}{\ln(n)} > \frac{1}{\ln(n+1)}$

AND $\lim_{n \to \infty} \frac{1}{\ln(n)} = 0$.

SO THE SERIES $\boxed{\text{CONVERGES}}$ BY THE ALT. SERIES TEST.

11. (20 points)Does the series $\sum_{n=1}^{\infty} \frac{n!}{(2n)!} 10^n$ converge of diverge? Explain why.

APPLYING THE RATIO TEST,

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)!/(2n+2)! \; 10^{n+1}}{n!/(2n)! \; 10^n}$$

$$= \lim_{n \to \infty} \frac{(n+1)}{(2n+2)(2n+1)} \cdot 10$$

$$= \lim_{n \to \infty} \frac{n+1}{4n^2 + \text{STUFF}} \cdot 10 = 0.$$

SINCE $0 < 1$, THE SERIES $\boxed{\text{CONVERGES}}$

BY THE RATIO TEST.

12. (20 points) Find the radius of convergence and the interval of convergence of the power series $f(x) = \sum_{n=0}^{\infty} \frac{n}{2^n}(x-1)^n$.

APPLY RATIO TEST:

$$\lim_{n\to\infty} \left| \frac{n+1}{2^{n+1}}(x-1)^{n+1} \cdot \frac{2^n}{n(x-1)^n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{1}{2} \cdot (x-1) \right|$$

$$= \left| \frac{x-1}{2} \right|.$$

WE WANT x SO THAT $\frac{1}{2}|x-1| < 1$

IE $|x-1| < 2$

$-2 < x-1 < 2$

$-1 < x < 3$.

RADIUS $= 2$

CENTER $= 1$.

WHEN $x = 3$, SERIES IS

$\sum \frac{n}{2^n} \cdot 2^n = \sum n$, WHICH DIVERGES ($\lim_{n\to\infty} n \neq 0$)

WHEN $x = -1$, SERIES IS

$\sum \frac{n}{2^n}(-2)^n = \sum (-1)^n n$, WHICH DIVERGES ($\lim_{n\to\infty} (-1)^n n \neq 0$)

THUS, INTERVAL OF CONVERGENCE IS

$\boxed{x \in (-1, 3)}$

13. (15 points) Use the Maclaurin series for $\sin(x)$ (which you should have memorized) to find the 10th degree Taylor polynomial for $\sin(x^2)$ at $a = 0$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$$

THUS,

$$\boxed{T_{10}(x) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}}$$

 NAME: SOLUTIONS SECTION:

14. (30 points) Find the Taylor series for the function $f(x) = \frac{1}{x}$ at $a = 1$. Do this in three different ways:

(a) From the general formula (without using any Taylor series which you have memorized)

0 $f(x) = x^{-1}$ $\quad f(1) = 1$

1 $f'(x) = -x^{-2}$ $\quad f'(1) = -1$

2 $f''(x) = 2x^{-3}$ $\quad f''(1) = 2$ $\quad f''(1)/2 = 1$

3 $f'''(x) = -3!x$ $\quad f'''(1) = -3!$ $\quad f'''(1)/3! = -1$

$\vdots$

$f^{(n)}(x) = (-1)^n n! x$ $\qquad \frac{f^{(n)}(1)}{n!} = (-1)^n$

$$\therefore f(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots = \boxed{\sum_{n=0}^{\infty} (x-1)^n (-1)^n} = \sum_{n=0}^{\infty} (1-x)^n$$

(b) Using the Taylor series: $ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$.

SINCE $\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$ TAKING THE DERIVATIVE

GIVES $\frac{1}{x} = \sum_{n=1}^{\infty} (-1)^{n+1}(x-1)^{n-1} = \boxed{\sum_{k=0}^{\infty} (-1)^k (x-1)^k}$

LET $k = n-1$ $\qquad = \sum_{k=0}^{\infty} (1-x)^k$

(c) Starting with the Maclaurin series for $\frac{1}{1-x}$ (which you should have memorized), and making a substitution.

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots = \sum_{n=0}^{\infty} r^n$$

LET $r = 1-x$, SO $1 - r = x$. THUS

$$\frac{1}{x} = 1 + (1-x) + (1-x)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} (1-x)^n$$

SOLUTIONS

15. (25 points) Newton's Law of cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings. I have just poured a cup of $100°F$ coffee in a room where the temperature is $50°F$. Let $f(t)$ denote the coffee temperature t hours after I poured it.

(a) Write a differential equation and initial condition that $f(t)$ satisfies.

$$\begin{cases} f'(t) = k\left(f(t) - 50\right) \\ f(0) = 100 \end{cases}$$

(b) Solve the initial value problem, assuming the coffee temperature is initially dropping at a rate of 40 degrees per hour (that is, $f'(0) = -40$).

$$\frac{df}{dt} = k(f - 50)$$

$$\int \frac{df}{f-50} = \int k\,dt$$

$$\ln|f-50| = kt + C$$

$$f - 50 = Ae^{kt}, \quad f = Ae^{kt} + 50$$

SINCE $f(0) = 100$, $100 = A + 50 \Rightarrow A = 50$.

$$f'(t) = 50ke^{kt}.$$

SINCE $f'(0) = -40$, $-40 = 50k$, so $k = -4/5$.

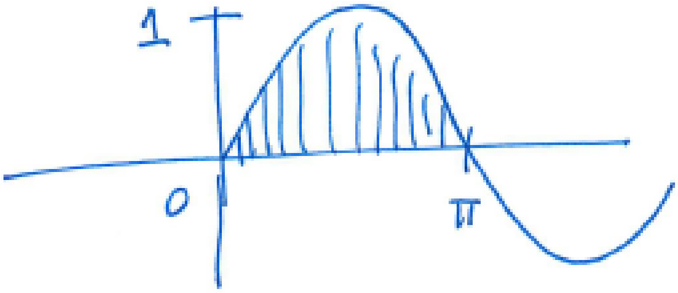
THUS

$$\boxed{f(t) = 50e^{-4t/5} + 50}$$

 NAME: SOLUTIONS. SECTION:

16. (25 points)

(a) Sketch a picture of the region above the x-axis, under the graph of $f(x) = \sin(x)$, and between $x = 0$ and $x = \pi$.

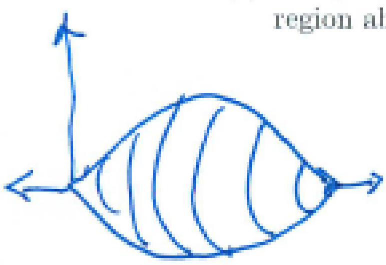


(b) Compute the area of this region.

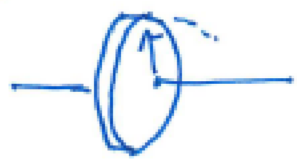
$$\int_0^{\pi} \sin(x)\,dx = -\cos x \Big|_0^{\pi} = 1 - (-1)$$

$$= \boxed{2}$$

(c) Compute the volume of the solid obtained by revolving this region about the x-axis.



AT A TYPICAL X VALUE, A CROSS SECTION IS A DISK



OF RADIUS $\sin x$, THICKNESS dx.

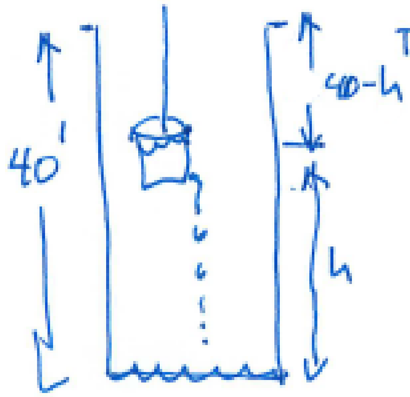
AREA $= \pi(\sin x)^2$.

THUS, VOLUME $= \int_0^{\pi} \text{AREA}\, dx$

$$\text{VOL} = \int_0^{\pi} \pi \sin^2 x\, dx = \pi \int_0^{\pi} \tfrac{1}{2}(1 - \cos 2x)\, dx$$

$$= \pi \left(\tfrac{1}{2}\right)\left(x - \tfrac{1}{2}\sin 2x\right)\Big|_0^{\pi} = \boxed{\frac{\pi^2}{2}}$$

17. (EXTRA CREDIT – 20 points) I lift water from a 40 foot deep well by means of a bucket attached to a rope. When the bucket is full of water, it weighs 30 pounds. But the bucket has a leak that causes it to lose water at a rate of $\frac{1}{4}$ pound for each foot that I raise the bucket. Neglecting the weight of the rope, find the work done (in foot-pounds) in raising the (initially full) bucket from the bottom of the well to the top of the well.



THE WORK DONE IS THE INTEGRAL OF THE FORCE OVER THE DISTANCE. LET'S LET h BE THE DISTANCE BETWEEN THE BOTTOM OF THE WELL AND THE BUCKET, (SO $0 < h < 40$), AND $w(h)$ BE THE WEIGHT OF THE BUCKET AT HEIGHT h.

THUS, $w(0) = 30$

$w(h) = 30 - \left(\frac{1}{4}\right)h$ [SINCE IT LOSES 1/4 lb PER FOOT]

THEN THE WORK IS

$$\int_0^{40} \left(30 - \frac{h}{4}\right) dh$$

$$= 30h - \frac{h^2}{8}\Big|_0^{40}$$

$$= 1200 - \frac{1600}{8} = 1000 \text{ FT-POUNDS}$$

Since the units are in pounds, which is a force, we don't need to multiply by g.