MATH 132 Solutions to Midterm 2 (papaya)

1. For each of the following sequences, determine whether it converges or diverges. If the sequence converges, give its limit. Justify your answer.

5 pts (a) $\left\{\dfrac{2n^4+\cos(n\pi)}{(1+3n)(2+n^3)}\right\}_{n=0}^{\infty} = -\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{10}, \ldots$

Solution: For n large, the $\cos(n\pi)$ term is irrelevant, as are the lower powers of n in the denominator. Thus

$$\lim_{n\to\infty} \frac{2n^4+\cos(n\pi)}{(1+3n)(2+n^3)} = \lim_{n\to\infty} \frac{2n^4}{3n^4} = \frac{2}{3}$$

5 pts (b) $\left\{\dfrac{\ln(n)}{2n}\right\}_{n=1}^{\infty} = 0, \dfrac{\ln 2}{4}, \dfrac{\ln 3}{6}, \ldots$

Solution: $\displaystyle\lim_{n\to\infty} \frac{\ln(n)}{2n}$ is of the form ∞/∞, so we can use L'Hôpital's rule. Thus,

$$\lim_{n\to\infty} \frac{\ln(n)}{2n} = \lim_{n\to\infty} \frac{1/n}{1} = 0$$

5 pts (c) $\{e+(-1)^n\}_{n=0}^{\infty} = e+1, e-1, e+1, \ldots$

Solution: The sequence diverges, because the limit as $n \to \infty$ does not exist; it alternates between $e+1$ and $e-1$.

10 pts 2. What value of N do we need to ensure that the sum $\displaystyle\sum_{n=0}^{N} \frac{(-1)^n}{n^2+2}$ is within $1/100$ of the limit? Keep in mind that N must be a whole number.

Solution: Since this is an alternating series, we know that the remainer R_N is less than the absolute value of the $N+1$-st term. This means we want N so that

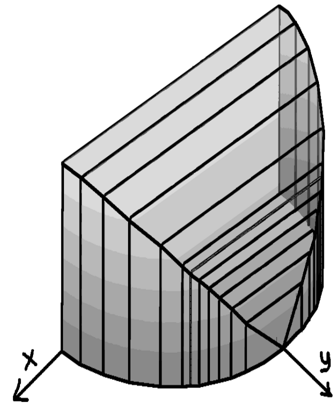
$$\frac{1}{(N+1)^2+2} < \frac{1}{100} \qquad \text{or, equivalently} \qquad 100 < (N+1)^2+2$$

This holds for $N = 9$.

$\boxed{\text{15 pts}}$ 3. Find the volume of the solid whose base is the half-circle

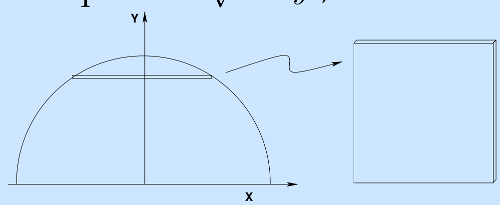
$$x^2 + y^2 = 9 \qquad \text{with } y \geq 0$$

and whose cross-sections perpendicular to the y-axis are squares.



Solution: Note that since the cross-sections perpendicular to the y-axis are squares, we want to integrate dy. (If we tried to integrate dx, we would have very complicated slices.)

So, we write the base as $x = \pm\sqrt{9 - y^2}$, and observe that if we take a slice at a particular y value, the square cross section will stretch from $x = -\sqrt{9 - y^2}$ to $x = +\sqrt{9 - y^2}$. This means the side length of the square is $2\sqrt{9 - y^2}$, and its area is $4(9 - y^2)$.

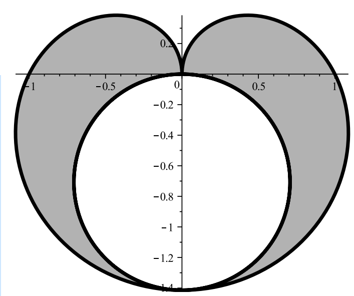


The volume of the solid is then given by integrating the cross-sectional area as y ranges from 0 to 3.

$$Vol = \int_0^3 4(9 - y^2)\,dy = 4(9y - y^3/3)\Big|_0^3 = 4(27 - \frac{27}{3}) = 72.$$

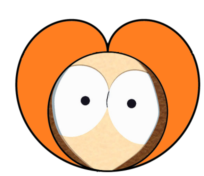
$\boxed{\text{15 pts}}$ 4. Find the area that lies inside the polar curve $r = \sqrt{1 - \sin\theta}$

but outside the circle $r = -\sqrt{2}\sin\theta$.



Solution: It is easiest to do this by calculating the outer area first, and then subtract off the inner area. The two curves meet at the origin and along the negative y-axis.
The area inside the cardiod-like shape is

$$\begin{aligned}\frac{1}{2}\int_0^{2\pi}\left(\sqrt{1 - \sin\theta}\right)^2 d\theta &= \frac{1}{2}\int_0^{2\pi} 1 - \sin\theta\,d\theta \\ &= \frac{1}{2}\Big(\theta + \cos\theta\Big)\Big|_0^{2\pi} = \frac{1}{2}\Big((2\pi + 1) - 1\Big) \\ &= \pi.\end{aligned}$$



The circle has radius $\sqrt{2}/2$, so its area is $\pi/2$.

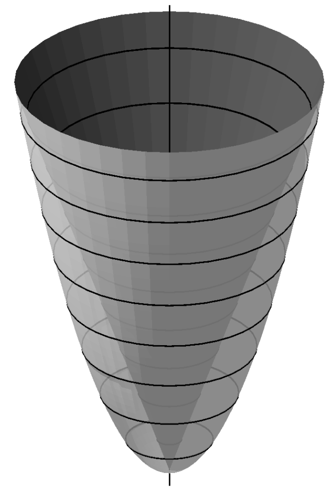
This means the shaded area is $\pi - \pi/2 = \dfrac{\pi}{2}$.

15 pts 5. Find the volume of the solid obtained by rotating the region between the two curves

$$y = 2x \qquad \text{and} \qquad y = x^2$$

about the y-axis.

(a) Write an integral which represents the volume.



Solution: Note that the curves cross at $(0,0)$ and $(2,4)$.

We can integrate dy or dx, as we wish. If we integrate with respect to x, we have cylindrical shells. The height of each cylinder runs from $y = x^2$ to $y = 2x$, so it is $2x - x^2$. The radius of the cylinder is x. This means the volume is given by

$$\int_0^2 2\pi x(2x - x^2)\,dx = 2\pi \int_0^2 2x^2 - x^3\,dx.$$

If instead we want to integrate dy, note that such a slice will be a "washer" with inner radius $x = y/2$ and outer radius $x = \sqrt{y}$. This means we have the integral

$$\int_0^4 \pi(\sqrt{y})^2 - \pi(y/2)^2\,dy = \pi \int_0^4 y - \frac{y^2}{4}\,dy$$

(b) Evaluate the integral in (a).

Solution: For the first integral, we have

$$2\pi \int_0^2 2x^2 - x^3\,dx = 2\pi \left(\frac{2}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^2 = 2\pi\left(\frac{16}{3} - \frac{16}{4}\right) = \frac{8\pi}{3}.$$

The second integral gives

$$\pi \int_0^4 y - \frac{y^2}{4}\,dy = \pi\left(\frac{y^2}{2} - \frac{y^3}{12}\right)\Big|_0^4 = \pi\left(\frac{16}{2} - \frac{64}{12}\right) = \frac{8\pi}{3}.$$

Of course, they both evaluate to the same value.

15 pts 6. Determine the interval of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(3x-1)^n}{\sqrt{n+5}}$. Don't forget to establish convergence or divergence at the endpoints!

Solution: First, we apply the ratio test, calculating $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(3x-1)^{n+1}}{\sqrt{n+1+5}} \cdot \frac{\sqrt{n+5}}{(-1)^n(3x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+5}}{\sqrt{n+6}} \cdot (3x-1)\right| = |3x-1|$$

Solution: (continued) So, for the series to converge, we must have $|3x-1|<1$, that is, $-1<3x-1<1$. Adding 1 to both sides yields $0<3x<2$, or equivalently $0<x<\frac{2}{3}$. (If you prefer, observe that the center is $\frac{1}{3}$ and the radius of convergence is also $\frac{1}{3}$.)

Now we need to establish what happens at the endpoints.

When $x=0$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{\sqrt{n+5}}$, or, equivalently, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+5}}$. This series diverges. If you like, you can use the integral test, or we can use limit comparison with a p-series with $p=1/2$:

$$\lim_{n\to\infty}\frac{1/\sqrt{n+5}}{1/\sqrt{n}}=1,$$

so the two series do the same thing. Since $\sum 1/\sqrt{n}$ diverges, so does the original.

When $x=2/3$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+5}}$. This is an alternating series. Since the terms are decreasing and $\lim 1/\sqrt{n+5}=0$, the series converges.

Consequently, the interval of convergence is $\left(0,\dfrac{2}{3}\right]$.

7. For each of the infinite sums below, state whether it converges or diverges. Justify your answer completely.

5 pts (a) $\displaystyle\sum_{n=1}^{\infty}\frac{\ln(n)}{2n}=0+\frac{\ln 2}{4}+\frac{\ln 3}{6}+\dots$

Solution: Diverges by the integral test, or by comparison with $\sum\frac{1}{2n}$ (The series is larger than a divergent series, and so diverges).

5 pts (b) $\displaystyle\sum_{n=3}^{\infty}\frac{n^2+5}{(n^2-1)(n^2-4)}=\frac{7}{20}+\frac{7}{60}+\frac{1}{16}+\dots$

Solution: Since $\dfrac{n^2+5}{(n^2-1)(n^2-4)}<\dfrac{n^2}{n^4}=\dfrac{1}{n^2}$, the series converges by comparison with a convergent p-series ($p=2$).

5 pts (c) $\displaystyle\sum_{n=5}^{\infty}\frac{5^n-3^n}{7^{n+2}}=\frac{2882}{82543}+\frac{304}{117649}+\frac{75938}{40353607}+\dots$

Solution: Observe that $\displaystyle\sum\frac{5^n-3^n}{7^{n+2}}=\frac{1}{49}\left(\sum\frac{5^n}{7^n}-\sum\frac{3^n}{7^n}\right)$. So this is the difference of two convergent geometric series (the ratios are both less than one), and it converges (to $\frac{1}{49}(\frac{7}{2}-\frac{7}{4})=1/28$).

5 pts (d) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{\sqrt{n^2+1}}=\frac{1}{\sqrt{2}}-\frac{2}{\sqrt{5}}+\frac{3}{\sqrt{10}}-\dots$

Solution: This diverges, since the limit of a_n is not zero.

15 pts

8. Jack has a goose that lays golden eggs, one each day. Unfortunately each egg is only $7/10$ the mass of the previous one. Jack needs to obtain 120 grams of gold to ransom his sister from the evil monkey-king. If the first egg weighed 30 grams, does Jack ever get enough gold? If so, how long must he wait? Justify your answer.

Solution: The total amount of gold (in grams) that Jack will get is

$$30 + 30 \cdot \frac{7}{10} + 30 \cdot \frac{7^2}{10^2} + 30 \cdot \frac{7^3}{10^3} + \ldots = 30 \sum_{n=0}^{\infty} \frac{7^n}{10^n}$$

This is a geometric series, and the sum is

$$\frac{30}{1 - 7/10} = \frac{30}{3/10} = 100$$

Sadly, no matter how long he waits, he will never even get 100 grams of gold.

15 pts

9. Recall that Hooke's law says that the amount of force required to stretch a spring x units beyond its natural length is kx, where k is a constant depending on the spring.

A giant spring designed to hold the gates of Mordor closed has a natural length of 10 meters. If it takes 1800 Joules[1] to stretch the spring from 10 meters to a length of 13 meters, how much work will it take to stretch the spring from 13 meters to a length of 16 meters?

Solution: First, we need to determine the spring constant. Since the amount of work to stretch the spring from 10 to 13 is 1800 J, we have

$$1800 = \int_0^3 kx\, dx = \left.\frac{kx^2}{2}\right|_0^3 = \frac{9k}{2},$$

and so $k = 1800 \cdot 2/9 = 400$. (We integrated from 0 to 3 because the natural length of the spring is 10 meters).

Now the required work to stretch the spring from 13 to 16 is given by

$$\int_3^6 400x\, dx = \left.\frac{400x^2}{2}\right|_3^6 = 200(36 - 9) = 200 \cdot 27 = 5400\ J$$

[1] A Joule is the amount of work needed to apply a force of 1 Newton over the distance of 1 meter.