MAT 126 Calculus B Spring 2007
Practice Midterm II Solutions

Answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer. The actual exam will contain 5 problems. This practice test contains more problems to give you more practice.

1. Evaluate the following definite integrals

(a)

$$\int_0^{13} \frac{2}{(2x+1)^{\frac{2}{3}}} dx$$

Solution. Substitution $u = 2x + 1$ gives $du = 2dx$ and $x = 0$ corresponds to $u = 1$ and $x = 13$ — to $u = 27$. Thus by the substitution rule,

$$\int_0^{13} \frac{2}{(2x+1)^{\frac{2}{3}}} dx = \int_1^{27} \frac{1}{u^{\frac{2}{3}}} du = 3u^{\frac{1}{3}}\Big|_1^{27}$$
$$= 3(27)^{\frac{1}{3}} - 3 = 3 \cdot 3 - 3 = 6.$$

(b)

$$\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x \, dx$$

Solution. Substitution $u = \sin x$ gives $du = \cos x dx$ and $x = 0$ corresponds to $u = 0$ and $x = \pi/2$ — to $u = 1$. Thus by the substitution rule,

$$\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x \, dx = \int_0^1 e^u du = e^u\big|_0^1 = e - 1.$$

(c)

$$\int_0^1 x^4(1+x^5)^{20} dx$$

Solution. Substitution $u = 1 + x^5$ gives $du = 5x^4 dx$ and $x = 0$ corresponds to $u = 1$ and $x = 1$ — to $u = 2$. Thus by the substitution rule,

$$\int_0^1 x^4(1+x^5)^{20} dx = \frac{1}{5}\int_1^2 u^{20} du = \frac{u^{21}}{5 \cdot 21}\Big|_1^2 = \frac{2^{21}-1}{105}.$$

(d)

$$\int_0^1 \tan^{-1} x\, dx$$

Solution. We use integration by parts with $u = \tan^{-1} x$ and $dv = dx$. We have $du = \dfrac{dx}{1+x^2}$ and $v = x$, so that using $\tan^{-1}(1) = \dfrac{\pi}{4}$, we get

$$\begin{aligned}\int_0^1 \tan^{-1} x\, dx &= \int_0^1 u dv = uv|_0^1 - \int_0^1 v du \\ &= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} dx.\end{aligned}$$

To evaluate the remaining integral, we use the substitution $u = 1 + x^2$, so that $du = 2xdx$ and $x = 0$ corresponds to $u = 1$ and $x = 1$ — to $u = 2$. Thus

$$\int_0^1 \frac{x}{1+x^2} dx = \frac{1}{2} \int_1^2 \frac{du}{u} = \left.\frac{\ln u}{2}\right|_1^2 = \frac{\ln 2}{2}.$$

Therefore, we get

$$\int_0^1 \tan^{-1} x\, dx = \frac{\pi}{4} - \frac{\ln 2}{2}.$$

(e)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin t|\, dt.$$

Solution. This is the integral over the symmetric interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ of the even function

$$f(t) = |\sin t|.$$

Since $\sin t$ is non-negative on $[0, \frac{\pi}{2}]$, we get

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin t|\, dt = 2 \int_0^{\frac{\pi}{2}} |\sin t|\, dt = 2 \int_0^{\frac{\pi}{2}} \sin t\, dt \\ = -2 \cos t|_0^{\frac{\pi}{2}} = -2(\cos \tfrac{\pi}{2} - \cos 0) = 2.\end{aligned}$$

(f)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+t^2)^2 \sin^5 t\, dt.$$

Solution. It is an integral over the symmetric interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ of the odd function

$$f(t) = (1+t^2)^2 \sin^5 t.$$

(Verify that $f(-t) = -f(t)$!). Therefore,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+t^2)^2 \sin^5 t \, dt = 0.$$

2. Evaluate the following indefinite integrals

(a)

$$\int x^3 e^{x^4} dx$$

Solution. Setting $u = x^4$, we get $du = 4x^3 dx$, so by the substitution rule,

$$\int x^3 e^{x^4} dx = \frac{1}{4} \int e^u du = \frac{1}{4} e^u + C = \frac{1}{4} e^{x^4} + C.$$

(b)

$$\int te^t dt$$

Solution. We use integration by parts with $u = t$ and $dv = e^t dt$. We have $du = dt$ and $v = e^t$, so that

$$\int te^t dt = \int udv = uv - \int vdu = te^t - \int e^t dt = te^t - e^t + C.$$

(c)

$$\int x^2 \cos x \, dx$$

Solution. We use integration by parts with $u = x^2$ and and $dv = \cos x dx$. We have $du = 2x dx$ and $v = \sin x$, so that

$$\int x^2 \cos x \, dx = \int udv = uv - \int vdu = x^2 \sin x - 2 \int x \sin x \, dx.$$

For the remaining integral we again use integration by parts with $u = 2x$ and $dv = \sin x dx$, so that $du = 2dx$ and $v = -\cos x$. We have

$$\begin{aligned} 2\int x \sin x \, dx &= \int udv = uv - \int vdu = -2x \cos x + 2 \int \cos x \, dx \\ &= -2x \cos x + 2 \sin x + C, \end{aligned}$$

so that

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

(Double-check the answer by differentiating!)

(d)
$$\int \cos(\sqrt{x})dx$$

Solution. First, we use the substitution rule with $t = \sqrt{x}$, so that $dt = \dfrac{1}{2\sqrt{x}}\,dx$, or $dx = 2\sqrt{x}\,dt = 2t\,dt$. We get

$$\int \cos(\sqrt{x})dx = 2\int t\cos t dt.$$

To evaluate this integral, we use integration by parts with $u = 2t$ and $dv = \cos t dt$. We have $du = 2dt$ and $v = \sin t$, so that

$$\begin{aligned} 2\int t\cos t dt &= \int u dv = uv - \int v du \\ &= 2t\sin t - 2\int \sin t dt = 2t\sin t + 2\cos t + C. \end{aligned}$$

Finally, remembering that $t = \sqrt{x}$, we get

$$\int \cos(\sqrt{x})dx = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C.$$

3. Evaluate the following indefinite integrals

(a)
$$\int \frac{1}{x^2}\ln x dx$$

Solution. Here we use integration by parts with $u = \ln x$ and $dv = \dfrac{1}{x^2}dx$, so that

$$du = \frac{1}{x}dx \quad \text{and} \quad v = -\frac{1}{x}.$$

(Note that substitution rule with $u = \ln x$ does not simplify the integral since in the denominator we have x^2; if it was x, then the substitution rule would work.) Thus we have

$$\begin{aligned} \int \frac{1}{x^2}\ln x dx &= \int u dv = uv - \int v du \\ &= -\frac{\ln x}{x} + \int \frac{1}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{x} + C. \end{aligned}$$

(b)
$$\int \frac{1}{x}(\ln x)^2 dx$$

Solution. Here we use the substitution rule with $u = \ln x$ and $du = \frac{1}{x}dx$ (since we have x in the denominator). Therefore,

$$\int \frac{1}{x}(\ln x)^2 dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C.$$

(c)

$$\int x^7 \ln x\, dx$$

Solution. Here we use integration by parts with $u = \ln x$ and $dv = x^7 dx$, so that

$$du = \frac{1}{x}dx \quad \text{and} \quad v = \frac{1}{8}x^8.$$

Thus we have

$$\begin{aligned}\int x^7 \ln x\, dx &= \int u dv = uv - \int v du \\ &= \frac{1}{8}x^8 \ln x - \frac{1}{8}\int x^8 \frac{dx}{x} = \frac{1}{8}x^8 \ln x - \frac{1}{64}x^8 + C.\end{aligned}$$

4. Evaluate the following indefinite integrals

(a)

$$\int \frac{2x^2}{x^2+1}dx$$

Solution. We have

$$\frac{2x^2}{x^2+1} = 2 - \frac{2}{x^2+1}$$

(either by doing the long division, or by writing $2x^2 = 2x^2 + 2 - 2 = 2(x^2+1) - 2$, and dividing both terms by x^2+1). Thus

$$\int \frac{2x^2}{x^2+1}dx = \int \left(2 - \frac{2}{x^2+1}\right) dx = 2x - 2\tan^{-1} x + C.$$

(b)

$$\int \frac{2x}{x^2+1}dx$$

Solution. To evaluate this integral, we use the substitution $u = 1 + x^2$, so that $du = 2xdx$ (compare with the last integral in problem **2** (d)). We have

$$\int \frac{2x}{x^2+1}dx = \int \frac{du}{u} = \ln|u| + C = \ln(1+x^2) + C.$$

5. (a) Write a formula for $\tan x$ in terms of $\sin x$ and $\cos x$.
Solution.
$$\tan x = \frac{\sin x}{\cos x}.$$
(b) Evaluate
$$\int \tan x \, dx$$
Solution. Using part (a) we have
$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx,$$
which suggests the substitution $u = \cos x$. We have $du = -\sin x dx$, so that
$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C.$$

6. Evaluate
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x}{\sin^2 x} \, dx$$
Solution. We use integration by parts with $= x$ and
$$dv = \frac{1}{\sin^2 x} \, dx,$$
so that $du = dx$ and $v = -\cot x$. We have, using that antiderivative of $\cot x$ is $\ln|\sin x|$ (compare with part (a)),
$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x}{\sin^2 x} \, dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} u dv = uv|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} v du \\ &= -x\cot x|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \, dx \\ &= \frac{\pi}{4} + \ln|\sin x|\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{4} + \frac{1}{2}\ln 2, \end{aligned}$$
where we have used that $\cot\frac{\pi}{4} = 1$, $\cot\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\sin\frac{\pi}{2} = 1$.

7. (a) Set
$$f(x) = \int_1^{x^2} \sin t^3 dt + x^3$$
Find $f(1)$ and $f'(x)$.
Solution. First, $f(1) = 1^3 = 1$, since in this case the interval of integration shrinks to a point and the integral

is zero. Second, we get by the FTC and the chain rule, setting $u = x^2$,

$$\begin{aligned} f'(x) &= \frac{du}{dx}\ \frac{d}{du}\left(\int_1^u \sin t^3 dt\right)\Big|_{u=x^2} + 3x^2 \\ &= 2x\ (\sin u^3)\big|_{u=x^2} + 3x^2 = 2x \sin x^6 + 3x^2. \end{aligned}$$

(b) Set

$$f(x) = \int_{\sqrt{x}}^{x-2} \tan^2 t\, dt$$

Find $f(4)$ and $f'(x)$

Solution. We have

$$f(4) = \int_{\sqrt{4}}^{4-2} \tan^2 t\, dt = \int_2^2 \tan^2 t\, dt = 0.$$

To find $f'(x)$, we write

$$\begin{aligned} f(x) &= \int_{\sqrt{x}}^{x-2} \tan^2 t\, dt = \int_{\sqrt{x}}^{0} \tan^2 t\, dt + \int_0^{x-2} \tan^2 t\, dt \\ &= -\int_0^{\sqrt{x}} \tan^2 t\, dt + \int_0^{x-2} \tan^2 t\, dt, \end{aligned}$$

and apply the FTC and the chain rule (for the first integral we use $= \sqrt{x}$, and for the second integral we use $u = x-2$). We get

$$\begin{aligned} f'(x) &= -\frac{du}{dx}\ \frac{d}{du}\left(\int_0^u \tan^2 t\, dt\right)\Big|_{u=\sqrt{x}} + \frac{du}{dx}\ \frac{d}{du}\left(\int_0^u \tan^2 t\, dt\right)\Big|_{u=x-2} \\ &= -\frac{1}{2\sqrt{x}} \tan^2(\sqrt{x}) + \tan^2(x-2). \end{aligned}$$