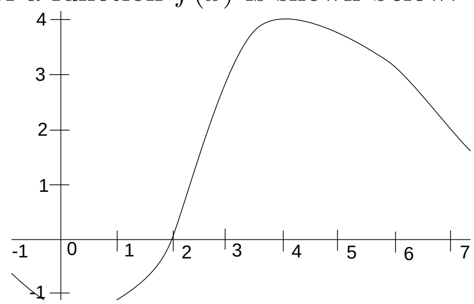
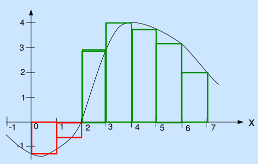
MAT 126: Calculus B
Solutions to Midterm 1, Spring 2000

1. (30 points) The graph of a function $f(x)$ is shown below.



(a) Estimate the area under the graph between $x = 2$ and $x = 7$.

Solution: We can do this in many ways, but most of them come down to doing some kind of Riemann sum. Let's do a right sum, with 5 rectangles.



We just add the areas of the five green rectangles together. The width of each is $(7-2)/5 = 1$, so our estimate is:

$$1\cdot\big(f(3)+f(4)+f(5)+f(6)+f(7)\big) = 1\cdot(3+4+4.5+3+2) = 16.5$$

It is important to realize that there is no single correct answer here. Essentially any answer between about 8 and 20 can be correct, as long as the reasoning for it is valid. Typically this reasoning will be using some kind of rectangles to cover the area.

(b) Estimate $\displaystyle\int_0^7 f(x)\ dx$

Solution: Here, we use the fact that $\displaystyle\int_0^7 f(x)\ dx = \int_0^2 f(x)\ dx + \int_2^7 f(x)\ dx$. We already estimated $\int_2^7 f(x)\ dx$ as 16.5 in the first part; we just need to estimate $\int_0^2 f(x)\ dx$. For variety, let's use two rectangles which cross the graph over their midpoints. This gives us

$$1\cdot f(\frac{1}{2}) + 1\cdot f(\frac{3}{2}) = -1.25 + (-.75) = -2$$

as an estimate for $\int_0^2 f(x)\ dx$.

This gives us an estimate of $-2 + 16.5 = 14.5$ for $\int_0^7 f(x)\ dx$. As in the first part, there is no single correct answer.

2. (20 points) A car is traveling back and forth along a straight road that runs East-West. We make the convention that positive velocity means velocity Eastward. During a half hour, we measure the velocity (in miles per hour) every 6 minutes (6 minutes = 0.1 hour), with the following results.

t in hours	0	0.1	0.2	0.3	0.4	0.5
velocity in m.p.h	20	30	20	-10	-10	0

Use the data in the table to estimate:

(a) Where the car is at the end of the half hour with respect to where it was at the beginning. Tell us explicitly how you are making the estimate!

Solution: If we let $v(t)$ represent the (signed) velocity of the car at time t, then we are being asked to estimate $\int_0^{0.5} v(t)\ dt$.

As in the previous question, we can do this using a Riemann sum. Let's use the left sum with 5 rectangles, L_5. Here, the width of each rectangle is 1/10.

$$\begin{aligned} L_5 =& \frac{1}{10}\Big(v(0) + v(.1) + v(.2) + v(.3) + v(.4)\Big) \\ =& \frac{1}{10}\Big(20 + 30 + 20 + (-10) + (-10)\Big) = 50/10 = 5 \end{aligned}$$

So our estimate is that after half an hour, the car is 5 miles from where it started. You might have chosen to use R_5, which would give an estimate of 3 miles. Or maybe you did something else reasonable, or something unreasonable.

(b) How far the car drove (total mileage) during the half hour. Tell us explicitly how you are making the estimate!

Solution: In this case, we are being asked to estimate $\int_0^{0.5} |v(t)|\ dt$.

Again, using left rectangles, we get

$$\begin{aligned} L_5 =& \frac{1}{10}\Big(|v(0)| + |v(.1)| + |v(.2)| + |v(.3)| + |v(.4)|\Big) \\ =& \frac{1}{10}\Big(20 + 30 + 20 + 10 + 10\Big) = 90/10 = 9. \end{aligned}$$

So my estimate is that the car traveled a total of 9 miles (7 forward, and 2 back).

3. (30 points) Find anti-derivatives of the following functions:

(a) $f(x) = x^2 - 2\cos(x)$

Solution:

$$\int x^2 - 2\cos(x)\ dx = \frac{x^3}{3} - 2\sin(x) + C$$

(b) $f(x) = (1/2)\sin(3x)$

Solution: Remember that $\frac{d}{dx}\cos(3x) = -3\sin(3x)$, so we have

$$\int \frac{1}{2}\sin(3x)\ dx = -\frac{1}{6}\cos(3x) + C$$

(c) $f(x) = \dfrac{5}{\sqrt{x}}$

Solution:

$$\int \frac{5}{\sqrt{x}}\ dx = \int 5x^{-1/2}\ dx = 10x^{1/2} + C.$$

4. (20 points) Calculate the following definite integrals using the "Evaluation Theorem." *You must use the Evaluation Theorem to get credit!*

(a) $\displaystyle\int_0^2 x^5\ dx$

Solution:

$$\int_0^2 x^5\ dx = \frac{1}{6}x^6\Big|_0^2 = \frac{1}{6}2^6 - \frac{1}{6}0^6 = \frac{64}{6} = \frac{32}{3}$$

(b) $\displaystyle\int_{-1}^1 \frac{3}{1+x^2}\ dx$

Solution:

$$\int_{-1}^1 \frac{3}{1+x^2}\ dx = 3\arctan(x)\Big|_{-1}^1 = 3\arctan(1) - 3\arctan(-1) = \frac{3\pi}{4} - \left(-\frac{3\pi}{4}\right) = \frac{3\pi}{2}$$