MATH 125 Solutions to First Midterm

1. Compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

3 points

(a) $\displaystyle\lim_{x\to 2} \frac{x^2-4}{5x(x-2)}$

Solution:

$$\lim_{x\to 2} \frac{(x-2)(x+2)}{5x(x-2)} = \lim_{x\to 2} \frac{(x+2)}{5x} = \frac{2+2}{10} = \frac{2}{5}.$$

3 points

(b) $\displaystyle\lim_{x\to\infty} 2\cos\left(\frac{\pi}{x}\right)$

Solution:

$$\lim_{x\to\infty} 4\cos(\pi/x) = 4\cos(0) = 4.$$

3 points

(c) $\displaystyle\lim_{x\to 4} \frac{x^2}{(x-4)^2}$

Solution: Note for x close to 4, the numerator is close to 16 while the denominator tends towards zero. Thus, the function becomes unbounded at 4. Note also that the denominator is always positive. Hence, the limit is $+\infty$.

2. More of the same: compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

3 points

(a) $\displaystyle\lim_{x\to\infty} \frac{x^2-9}{5x(x-3)}$

Solution: For x very large, $x^2-9 \approx x^2$, and $x-3 \approx x$. Thus

$$\lim_{x\to\infty} \frac{x^2-9}{5x(x-3)} = \lim_{x\to\infty} \frac{x^2}{5x(x)} = \lim_{x\to\infty} \frac{1}{5} = \frac{1}{5}$$

3 points

(b) $\displaystyle\lim_{h\to 2} \frac{(x+h)^2-x^2}{h}$

Solution:

$$\lim_{h\to 2} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 2} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 2} \frac{2xh+h^2}{h} = \lim_{h\to 2} 2x+h = 2x+2.$$

3 points (c) $\lim_{x\to-\infty} e^x \cos(x)$

Solution: Observe that for any x, we have $-1 \le \cos(x) \le 1$, and so we also have $-e^x \le e^x \cos(x) \le e^x$. Applying the squeeze theorem,

$$\lim_{x\to-\infty}(-e^x) \le \lim_{x\to-\infty} e^x \cos(x) \le \lim_{x\to-\infty}(e^x),$$

that is,

$$0 \le \lim_{x\to-\infty} e^x \cos(x) \le 0.$$

Hence, the limit is 0.

3. Let $f(x) = 5x^3 - 8x + 2$.

3 points (a) Find the slope of the secant line passing through the points on the curve $y = f(x)$ where $x = 0$ and $x = 1$.

Solution: The slope of a line is the ratio of the change in y to the change in x. Here we have

$$\text{slope} = \frac{f(1) - f(0)}{1 - 0} = \frac{-1-2}{1} = -3.$$

3 points (b) Find $f'(1)$.

Solution: Using the power rule, $f'(x) = 15x^2 - 8$, so $f'(1) = 7$.

3 points (c) Write the equation of the tangent line to the graph of $y = f(x)$ when $x = 1$.

Solution: The point $(1, f(1))$ is on both the curve and the line. Now, $f(1) = 5 - 8 = -3$. We just need the equation of the line of slope 7 passing through the point $(1, -3)$. This is

$$y + 3 = 7(x-1) \qquad \text{or} \qquad y = 7x - 10.$$

3 points (d) At $x = 1$, is $f(x)$ concave up, concave down, or neither? Justify your answer fully.

Solution: Since $f''(x) = 30x$, we know $f''(1) > 0$. Thus $f(x)$ is concave up at $x = 1$.

8 points 4. For what values of x is the function $f(x) = \dfrac{e^x}{2 - e^{1/x}}$ continuous?

Solution: Since $f(x)$ is a composition of exponentials and rational functions, it is continuous everywhere on its domain.

Since $1/x$ is not defined for $x = 0$, the function is not continuous there.

Furthermore, there will be a discontinuity when the denominator is zero. That is, where $2 - e^{1/x} = 0$, or

$$\begin{aligned} 2 &= e^{1/x} \\ \ln(2) &= \ln\left(e^{1/x}\right) = 1/x \\ x &= \frac{1}{\ln(2)}. \end{aligned}$$

Thus, $f(x)$ is continuous at all real numbers except $x = 0$ and $x = \frac{1}{\ln(2)}$.

8 points 5. Write a limit that represents the slope of the graph

$$y = \begin{cases} |x|^x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

at $x = 0$. You **do not need to evaluate the limit.**

Solution: We just use the definition of the derivative at $x = 0$:

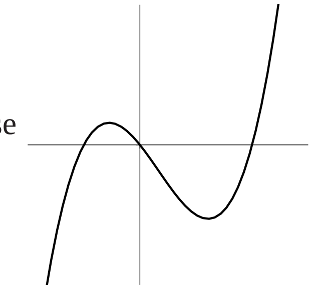
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}.$$

Since h is not zero, $f(h) = |h|^h$ and $f(0) = 1$. So,

$$f'(0) = \lim_{h \to 0} \frac{|h|^h - 1}{h}.$$

If you prefer to use the version of the definition $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$, you get the same answer except with x instead of h.

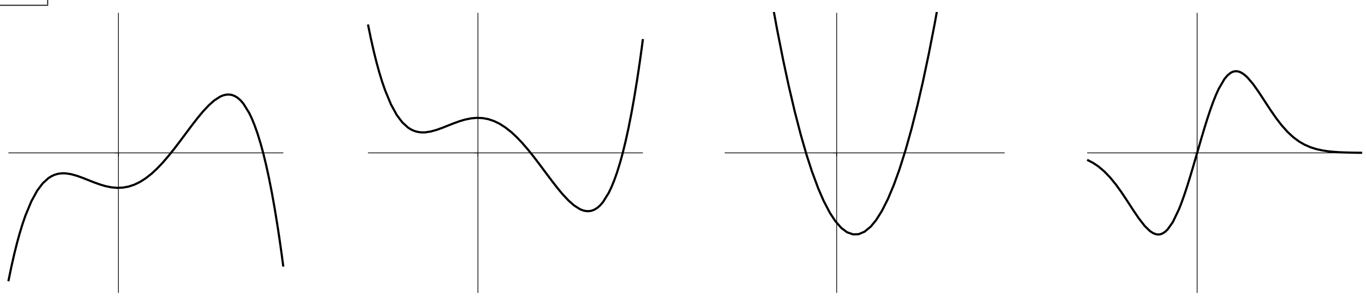
6. At right is the graph of **the derivative** $f'(x)$ of a function $f(x)$. Use it to answer each of the following questions.

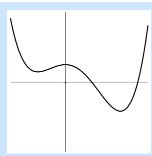


4 points (a) Is $f(x)$ concave up, concave down, or neither at $x = 0$?

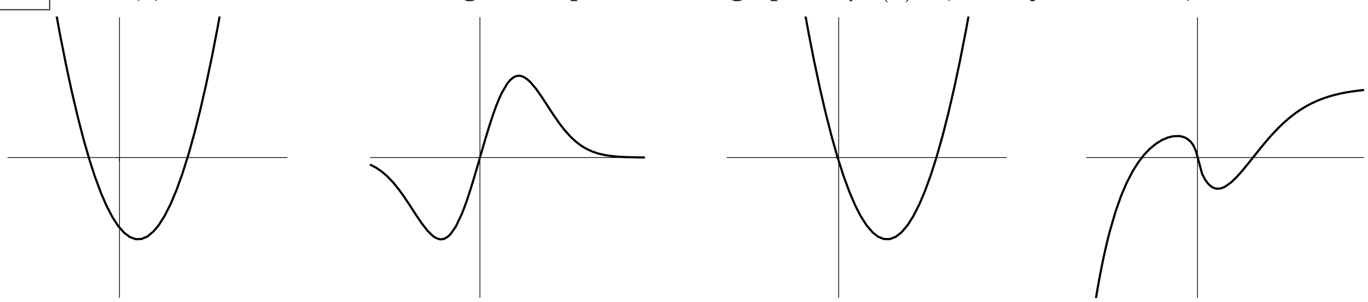
Solution: Since the derivative is decreasing at $x = 0$, we know $f(x)$ is concave down there.

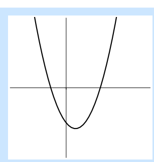
4 points (b) Which of the following best represents the graph of $f(x)$? (circle your answer).



Solution: The graph of $f(x)$ is 

4 points (c) Which of the following best represents the graph of $f''(x)$? (circle your answer).



Solution: The graph of $f''(x)$ is 

7. Let $f(x) = \dfrac{x^2 - 2x}{5(x^2-4)}$

4 points

(a) Identify the horizontal asymptotes of $f(x)$. If there are none, write "NONE".

Solution: A function $f(x)$ has a horizontal asymptote at $y = L$ when $\lim_{x\to\infty} f(x) = L$. So, we have

$$\lim_{x\to\infty} \frac{x^2-2x}{5(x^2-4)} = \lim_{x\to\infty} \frac{x^2}{5x^2} = \frac{1}{5}.$$

Thus, there is a horizontal asymptote $y = \frac{1}{5}$.

4 points

(b) Identify the vertical asymptotes of $f(x)$. If there are none, write "NONE".

Solution: We have a vertical asymptote at $x = a$ whenever $\lim_{x\to a^\pm} f(x) = \pm\infty$. The denominator of $f(x)$ factors as $5(x-2)(x+2)$, so we have to look at $a = 2$ and $a = -2$. Note that if $x \neq 2$ $x \neq -2$, we have

$$f(x) = \frac{x^2-2x}{5(x^2-4)} = \frac{x(x-2}{5(x-2)(x+2)} = \frac{x}{5(x+2)}$$

Near $x = -2$, we have

$$\lim_{x\to -2^+} f(x) = +\infty \qquad \text{and} \qquad \lim_{x\to -2^-} f(x) = -\infty,$$

so there is a vertical asymptote at $x = -2$.

Near $x = 2$, we have

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x}{5(x+2)} = \frac{2}{5(2+2)} = \frac{1}{10}.$$

Thus, $x = 2$ is not a vertical asymptote.

8 points

8. Write a function which expresses the area of a rectangle with a perimeter of 10 feet in terms of its width.

Solution: Let's let W denote the width of the rectangle (in feet), and L denote its length. Since the perimeter is 10, we know that

$$2L + 2W = 10,$$

or equivalently, $L = 5 - W$.

Since the area of the rectangle is LW, we have

$$A(W) = (5-W)W$$