Solutions for MAT 125 First Midterm

February 23, 2009

1. Let $f(x) = x^2 + 3x$ with domain all real numbers. Let $A = (1, f(1))$ and $B = (2, f(2))$. There is also the point $C = (x, f(x))$ with x close to 1.

(a) Calculate the slope of the line through A and B.

Solution. The line through two points (x_1, y_1) and (x_2, y_2) has slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

In this case take

$$(x_1, x_2) = A = (1, f(1)) = (1, 4)$$

and

$$(x_2, y_2) = B = (2, f(2)) = (2, 10).$$

This gives

$$m = \frac{10 - 4}{2 - 1} = 6.$$

(b) Give an equation for the line through A and B.

Solution. An equation for the line with slope m which contains a point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

By part (a) we know that the slope is $m = 6$. Taking $(x_1, y_1) = A = (1, 4)$ gives the equation

$$y - 4 = 6(x - 1)$$

which can be simplified to

$$y - 6x + 2 = 0.$$

(c) Explain that the slope of the line through A and C is given by

$$\text{slope} = \frac{x^2 + 3x - 4}{x - 1}.$$

Solution. By the same reasoning used in part (a), the slope of the line through $A = (1, 4)$ and $C = (x, f(x))$ is

$$\text{slope} = \frac{f(x) - 4}{x - 1} = x^2 + 3x - 4x - 1.$$

(d) Calculate the slope of the tangent line to the graph of f at A.

Solution. The slope of the tangent line to the graph of f at A is the limit as C approaches A of the slope of the line through A and C. As C approaches A, x approaches 1. Using the result of (c), we can write the slope of the tangent line to the graph of f at A as

$$\text{slope} = \lim_{x\to 1} \frac{x^2+3x-4}{x-1}.$$

To calculate this limit we use the factorization

$$x^2+3x-4 = (x+4)(x-1).$$

Now we can calculate the limit:

$$\begin{aligned} \text{slope} &= \lim_{x\to 1} \frac{x^2+3x-4}{x-1} \\ &= \lim_{x\to 1} \frac{(x+4)(x-1)}{x-1} \\ &= \lim_{x\to 1}(x+4) \\ &= 1+4 \\ &= 5. \end{aligned}$$

2.

(a) Calculate the limit

$$\lim_{x\to 2} \frac{3x^2-15x+18}{x-2}.$$

Solution. Observe that we can factor the numerator as

$$3x^2-15x+18 = 3(x^2-5x+6) = 3(x-2)(x-3).$$

This allows us to calculate the limit:

$$\begin{aligned} \lim_{x\to 2} \frac{3x^2-15x+18}{x-2} &= \lim_{x\to 2} \frac{3(x-2)(x-3)}{x-2} \\ &= \lim_{x\to 2} 3(x-3) \\ &= 3(2-3) \\ &= -1. \end{aligned}$$

(b) Calculate the limit

$$\lim_{x\to 2} \frac{3x^2-15x+19}{x-2}.$$

Solution. This limit does not exist (even as an infinite limit). First note that

$$\lim_{x\to 2^-} \frac{1}{x-2} = -\infty, \quad \text{and} \quad \lim_{x\to 2^+} \frac{1}{x-2} = +\infty.$$

Since $\lim_{x\to 2}(3x^2 - 15x + 19) = 1$, the limit laws (which are valid for infinite limits) tell us that

$$\begin{aligned}
\lim_{x\to 2^-} \frac{3x^2-15x+19}{x-2} &= \left(\lim_{x\to 2^-} (3x^2-15x+19)\right)\left(\lim_{x\to 2^-} \frac{1}{x-2}\right) \\
&= \lim_{x\to 2^-} \frac{1}{x-2} \\
&= -\infty.
\end{aligned}$$

The analogous calculation shows that

$$\lim_{x\to 2^+} \frac{3x^2-15x+19}{x-2} = +\infty.$$

Since the left limit is not equal to the right limit, we conclude that the limit does not exist.

3. Explain whether the function

$$f(x) = \begin{cases} \frac{x^2-3x}{x^2-9} & x \neq 3 \\ 21 & x = 3 \end{cases}$$

is continuous at $x = 3$ or not.

Solution. The function is continuous at $x = 3$ if and only if $\lim_{x\to 3} f(x) = f(3)$. But

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2-3x}{x^2-9} &= \lim_{x\to 3} \frac{x(x-3)}{(x-3)(x+3)} \\
&= \lim_{x\to 3} \frac{x}{x+3} \\
&= \frac{3}{6} \\
&= \frac{1}{2}.
\end{aligned}$$

Therefore the value of the limit is different from $f(3) = 21$, so the function is not continuous at $x = 3$.

4. Given the function

$$f(x) = \left[\frac{1}{1-x} + \frac{1}{x-3}\right] + \cos(\pi x),$$

with domain the numbers between 1 and 3, $1 < x < 3$.

(a) Calculate $f(2)$.

Solution. Since $\cos(2\pi) = 1$,

$$f(2) = \left[\frac{1}{2-1} + \frac{1}{2-3}\right] + \cos(2\pi) = [1-1] + 1 = 0 + 1 = 1.$$

(b) Is there a solution, a number x between 1 and 3, of $f(x) = 0$?

Solution. Yes. First note that

$$\begin{aligned} f(5/2) &= \left[\frac{1}{5/2-1} + \frac{1}{5/2-3}\right] + \cos(5\pi/2) \\ &= \left[\frac{1}{3/2} + \frac{1}{-1/2}\right] + 0 \\ &= \frac{2}{3} - 2 \\ &= -\frac{4}{3}. \end{aligned}$$

The function f is continuous on the closed interval $[2, 5/2]$ and satisfies $f(2) > 0$, $f(5/2) < 0$. By the intermediate value theorem there exists a number $x \in (2, 5/2)$ with $f(x) = 0$.

5. Calculate

$$\lim_{x\to\infty} \frac{3x^2+21}{7x^4+31x}.$$

Solution. First write

$$\frac{3x^2+21}{7x^4+31x} = \frac{3x^2+21}{7x^4+31x} \cdot \frac{1/x^4}{1/x^4} = \frac{3/x^2+21/x^4}{7+31/x^3}.$$

Using the limit laws and the fact that

$$\lim_{x\to\infty} \frac{1}{x^n} = 0$$

for any positive integer n, we get

$$\begin{aligned} \lim_{x\to\infty} \frac{3x^2+21}{7x^4+31x} &= \lim_{x\to\infty} \frac{3/x^2+21/x^4}{7+31/x^3} \\ &= \frac{3\lim_{x\to\infty}(1/x) + 21\lim_{x\to\infty}(1/x^4)}{7+31\lim_{x\to\infty}(1/x^3)} \\ &= \frac{3\cdot 0 + 21\cdot 0}{7+31\cdot 0} \\ &= 0. \end{aligned}$$

6.

(a) Calculate

$$\lim_{x\to 0^+} e^{-1/x}.$$

Solution. If $x > 0$ then $-1/x < 0$, and

$$\lim_{x\to 0^+} (-1/x) = -\infty.$$

By the law for limits of compositions,

$$\lim_{x\to 0^+} e^{-1/x} = \lim_{y\to -\infty} e^y = 0.$$

(b) Calculate

$$\lim_{x\to 0^-} e^{-1/x}.$$

Solution. If $x < 0$ then $-1/x > 0$ and

$$\lim_{x\to 0^-} (-1/x) = +\infty.$$

By the law for limits of compositions,

$$\lim_{x\to 0^-} e^{-1/x} = \lim_{y\to +\infty} e^y = +\infty.$$

7. Explain in words

$$\lim_{x\to \infty} f(x) = L.$$

Solution. This means that the values of the function $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

8. Sketch the graph of an example of a function f which satisfies all of the following conditions.

- $f(0) = 0$
- $f(7) = 11$
- $\lim_{x \to 7^-} f(x) = 3$
- $\lim_{x \to 7^+} f(x) = -3$
- $\lim_{x \to \infty} f(x) = 0$
- $\lim_{x \to 2^+} f(x) = \infty$
- $\lim_{x \to 2^-} f(x) = -\infty$
- $\lim_{x \to 1^+} f(x) = \infty$
- $\lim_{x \to -\infty} f(x) = -\infty$
- $f(1) = 3$
- $f(2) = 3$

Solution: One such graph is shown below. Other choices are possible, some are right, some are wrong.