Math 125 - Fall 2006
Solutions to the Practice Final Examination

1. Let f be a continuous function. Find

$$\lim_{x\to\infty} f\left((1-\tfrac{1}{x})^x\right).$$

Solution: Since f is continuous, we have

$$\lim_{x\to\infty} f\left((1-\tfrac{1}{x})^x\right) = f\left(\lim_{x\to\infty}(1-\tfrac{1}{x})^x\right).$$

Now,

$$\ln(1-1/x)^x = x\ln(1-1/x) = \frac{\ln[(x-1)/(x)]}{1/x} = \frac{\ln(x-1)-\ln(x)}{1/x}.$$

Using L'Hospital's rule, we have

$$\lim_{x\to\infty}\ln(1-1/x)^x = \lim_{x\to\infty}\frac{\frac{1}{x-1}-\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to\infty} -x^2\frac{x-(x-1)}{x(x-1)} = \lim_{x\to\infty} -\frac{x}{x-1} = -1.$$

Thus $\lim_{x\to\infty}(1-\frac{1}{x})^x = e^{-1}$, and so the answer is $f(e^{-1})$.

2. Consider the equation $x+e^x=0$. Is there a solution to this equation? Why or why not.

   Solution: Let $f(x)=x+e^x$. Then $f(-1)=-1+1/e<0$ while $f(0)=1>0$. By the Intermediate Value Theorem, $f(x)=0$ has a solution between 0 and -1.

3. Find the derivative of the function

$$e^{2\tan(\sqrt{x})}.$$

Solution:

$$\frac{d}{dx}e^{2\tan(\sqrt{x})} = e^{2\tan(\sqrt{x})}2(\sec(\sqrt{x}))^2\times\left(\frac{1}{2\sqrt{x}}\right).$$

4. Consider the function

$$f(x)=\begin{cases}\frac{\sin x}{x} & x<0\\ x^3+2x+1 & x\geq 0\end{cases}$$

At which points is f continuous? At which points is it differentiable?

Solution: The only point where we have to worry is $x=0$. Note that

$$\lim_{x\to 0^-} f(x)=1 \quad \text{while} \quad \lim_{x\to 0^+} f(x)=1.$$

Thus f is continuous at zero, and hence it is continuous everywhere.

Next,

$$f'(x) = \begin{cases} \frac{x\cos x - \sin x}{x^2} & x < 0 \\ \\ 3x^2 + 2 & x \geq 0 \end{cases}$$

Thus

$$\lim_{x \to 0^+} f'(x) = 2$$

while

$$\begin{aligned} \lim_{x \to 0^-} f'(x) &= \lim_{x \to 0^-} x\sin x + \cos x - \cos x 2x \\ &= \lim_{x \to 0^-} x\sin x + \cos x - \cos x 2x \\ &= \lim_{x \to 0^-} x \sin x 2x \\ &= \lim_{x \to 0^-} \sin x 2 = 0. \end{aligned}$$

Thus f is not differentiable at $x = 0$.

5. Let $f(x) = x \ln\left(1 + e^{x^2}\right)$. Find $f'(5)$.

   Solution:

$$f'(x) = \ln\left(1 + e^{x^2}\right) + \frac{x}{1 + e^{x^2}} \cdot e^{x^2} \cdot 2x.$$

   Thus $f'(5) = \ln(1 + e^{25}) + \frac{5}{1+e^{25}} \cdot e^{25} \cdot 10$.

6. Show that the curves

$$e^{x^2-y^2}\cos(2xy) = 1 \quad \text{and} \quad e^{x^2-y^2}\sin(2xy) = 0$$

   meet orthogonally at the point $(\sqrt{\pi}, \sqrt{\pi})$.

   Solution: Let y_1 and y_2 be the functions determined by the first and second equations respectively. By implicit differentiation, we have

$$e^{x^2-y_1^2}\cos(2xy_1)(2x - 2y_1y_1') - e^{x^2-y_1^2}\sin(2xy_1)(2y_1 + 2xy_1') = 0.$$

   Thus at $x = \sqrt{\pi}$ and $y_1 = \sqrt{\pi}$ we have

$$0 = e^{\pi-\pi}\cos(2\pi)(2\sqrt{\pi} - 2\sqrt{\pi}y_1') - e^{\pi-\pi}\sin(2\pi)(2\sqrt{\pi} + 2\sqrt{\pi}y_1') = 2\sqrt{\pi}(1 - y_1').$$

   Thus $y_1' = 1$.

   Similarly

$$e^{x^2-y_2^2}\sin(2xy_2)(2x - 2y_2y_2') + e^{x^2-y_2^2}\cos(2xy_2)(2y_2 + 2xy_2') = 0.$$

Thus at $x = \sqrt{\pi}$ and $y_1 = \sqrt{\pi}$ we have

$$0 = e^{\pi-\pi}\sin(2\pi)(2\sqrt{\pi} - 2\sqrt{\pi}y_2') - e^{\pi-\pi}\cos(2\pi)(2\sqrt{\pi} + 2\sqrt{\pi}y_2') = 2\sqrt{\pi}(1 + y_2').$$

Thus $y_2' = -1$. This means the slopes of the two tangent lines to the respective curves are reciprocal, so that the lines are indeed orthogonal, as desired.

7. Find the derivative of the function

$$f(x) = \frac{(\sin x)^2(\tan x)^2}{(x^2+1)^2}.$$

Solution: take $\ln$ of both sides to get

$$\ln f(x) = 2\ln\sin x + 2\ln\tan x - 2\ln(x^2+1).$$

Then

$$\frac{f'(x)}{f(x)} = 2\frac{\cos(x)}{\sin x} + 2\frac{(\sec(x))^2}{\tan x} - \frac{4x}{x^2+1}.$$

Thus

$$f'(x) = \left(2\frac{\cos(x)}{\sin x} + 2\frac{(\sec(x))^2}{\tan x} - \frac{4x}{x^2+1}\right)\left(\frac{(\sin x)^2(\tan x)^2}{(x^2+1)^2}\right).$$

8. Find an equation for the tangent line to the curve

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$

through the point $(0, 0.5)$.

Solution: By implicit differentiation, we have

$$2x + 2yy' = 2(2x^2 + 2y^2 - x)(4x - 4yy' - 1),$$

so at $x = 0, y = 0.5$ we have

$$y' = 2(0.5)(2y' - 1).$$

Solving, we get $y' = 1$, so the equation for the tangent line is

$$\frac{Y - 0.5}{X} = 1.$$

9. If $f(x) = e^x/(x+1)^3$, find $f'(x)$ and $f''(x)$.

Solution:

$$\ln f(x) = x - 3\ln(x+1) \Rightarrow f'(x) = (1 - 3/x)f(x) = (1 - 3/x)e^x/(x+1)^3.$$

Then differentiating $f'(x) = (1 - 3/x)f(x)$ we have

$$f''(x) = \frac{3}{x^2}f(x) + (1 - \tfrac{3}{x})f'(x) = \left(\tfrac{3}{x^2} + (1 - \tfrac{3}{x})^2\right)f(x) = \left(\tfrac{3}{x^2} + (1 - \tfrac{3}{x})^2\right)e^x/(x+1)^3.$$

10. Find the limit

$$\lim_{x\to 1} \frac{x^\pi - 1}{x^e - 1}.$$

Solution: This is the indeterminate form $0/0$. By L'Hospital's rule we have

$$\lim_{x\to 1} \frac{x^\pi - 1}{x^e - 1} = \lim_{x\to 1} \frac{\pi x^{\pi-1}}{e x^{e-1}} = \frac{\pi}{e}.$$

11. Show that $e^x \geq 1 + x$ for $x \geq 0$. (Hint: Consider the function $f(x) = e^x - 1 - x$.)

Solution: We are trying to show that $f(x) \geq 0$ for $x \geq 0$. Now, $f(0) = 1 - 1 - 0 = 0$, and $f'(x) = e^x - 1 \geq 0$ for $x \geq 0$. By the mean value theorem, there is some $c \in [0, x]$ so that

$$\frac{f(x) - f(0)}{x - 0} = f'(c) \geq 0.$$

It follows that $f(x) \geq f(0) = 0$ for $x \geq 0$, as desired.

12. A particle is moving along the curve $y = x^2$. As it passes through the point $(2, 4)$, its y coordinate changes at a rate of 5 m/sec. What is the rate of change of the particle's distance to the origin at this instant?

Solution: The distance D to the origin is

$$D = \sqrt{x^2 + y^2} = \sqrt{x^2 + x^4}.$$

Thus

$$\frac{dD}{dt} = \frac{2x + 4x^3}{2\sqrt{x^2 + x^4}} \frac{dx}{dt}.$$

Now $\frac{dy}{dt} = 2x\frac{dx}{dt}$. Thus

$$\frac{dD}{dt} = \frac{1 + 2x^2}{2\sqrt{x^2 + x^4}} 2x \frac{dx}{dt} = \frac{1 + 2x^2}{2\sqrt{x^2 + x^4}} \frac{dy}{dt}.$$

At $x = 2$ and $y = 4$ we have

$$\frac{dD}{dt} = \frac{9}{2\sqrt{20}} 5 = \frac{45}{4\sqrt{5}} m/sec.$$

13. Find the absolute maximum and absolute minimum values of the function

$$f(x) = x^2 - \ln x^2$$

on the interval $[1/4, 4]$.

Solution: $f'(x) = 2x - 2/x$, which is equal to zero when $x = -1$ and $x = +1$. Since we are only concerned with $\frac{1}{4} \leq x \leq 4$, we only need consider $x = 1$. Now we check the value of the endpoints and of the critical number, and we have:

$$f(1/4) = \frac{1}{16} + \ln(16) \qquad f(1) = 1 \qquad f(4) = 16 - \ln(16)$$

Thus, the absolute maximum occurs at $x = 4$ and the absolute minimum at $x = 1$. To answer the question,

$$f_{max} = 16 - \ln 16 \qquad f_{min} = 1.$$

14. Find

$$\lim_{x \to \frac{\pi}{2}} \tan(7x)\cos(4x)$$

Solution: The limit does not exist, since

$$\lim_{x \to \frac{\pi}{2}^+} \tan(7x) = +\infty, \qquad \lim_{x \to \frac{\pi}{2}^-} \tan(7x) = -\infty, \quad \text{and} \quad \lim_{x \to \frac{\pi}{2}} \cos(4x) = 1.$$

Thus, the limit from the left is $+\infty$ and from the right is $-\infty$, and the two sided limit does not exist.

15. A woman wants to get from a point A on the shore of a circular lake to a point C diametrically opposite A in the shortest possible time. She can walk at a speed of 4 mi/hr and row at a speed of 2 mi/hr. How should she proceed?

Solution: These are the strategies: walk half way around the circular lake, row straight across, or row to some point on the shore and walk from there. Select the one that minimizes the time it takes to get to point C.

Assume that the lake has radius 1 (all the times will multiply by a same factor of 'radius' if another radius is used, and it won't make a difference to determine which method is the faster). Represent the lake as the circle of radius 1 with the point C at angle $\theta = 0$ (the point (1,0)) and the point A at angle $\theta = \pi$ (the point (0,1)). Consider rowing to a point at angle $\theta = (cos(\theta), sin(\theta))$ with θ between 0 and π and walking along the circle the rest of the way to point C.

Notice that rowing to the point at angle π means not rowing at all, while rowing to the point at angle 0 mean only rowing, and no walking.

The rowing distance is equal to the distance between $(cos(\theta), sin(\theta))$ and the point $(-1, 0)$:

$$\sqrt{(cos(\theta) + 1)^2 + sin(\theta)^2} = \sqrt{2 + 2cos(\theta)} = \sqrt{2}\sqrt{1 + cos(\theta)}$$

The walking distance is equal to the distance between $(cos(\theta), sin(\theta))$ and the point $(1, 0)$ and is equal to θ.

The function to minimize is the total time (distance over velocity):

$$\begin{aligned} T(\theta) &= rowing - time + walking - time \\ &= \frac{\sqrt{2}}{2}\sqrt{1+cos(\theta)} + \frac{\theta}{4} \end{aligned}$$

For the endpoints:

- $\theta = 0$, only rowing, $T(0) = 1\,hr$.
- $\theta = \pi$, only walking, $T(\pi) = \frac{\pi}{4} = .785\,hr$, maybe the best strategy.

We need to check the alternatives. Let's find the critical numbers of T and check them. We obtain:

$$T'(\theta) = \frac{1}{2\sqrt{2}}\frac{-sin(\theta)}{\sqrt{1+cos(\theta)}} + \frac{1}{4}$$

Solve for the critical numbers:

$$\sqrt{2}sin(\theta) = \sqrt{1+cos(\theta)}$$

square and use $sin^2 = 1 - cos^2$:

$$2(1 - cos^2(\theta)) = 1 + cos(\theta)$$

which writes as: $2(1-cos(\theta))(1+cos(\theta)) = 1 + cos(\theta)$

- $1 + cos(\theta) = 0$, so $\theta = \pi$, a solution we already have.
- $2(1 - cos(\theta)) = 1$, i.e. $cos(\theta) = \frac{1}{2}$ i.e. $\theta = \frac{\pi}{3}$. In this case

$$T(\frac{\pi}{3}) = \frac{\sqrt{1+\frac{1}{2}}}{\sqrt{2}} + \frac{\pi}{12} = 1.12hr$$

So: she should just walk around the lake.

16. Consider the function $f(x) = x^3 - 7x^2 + 9x - \pi$.

- Find all the critical points of f. State whether each is a local minimum, local maximum, or neither.

  Solution: $f'(x) = 3x^2 - 14x + 9$. This is zero when

  $$x = \frac{14 \pm \sqrt{88}}{6} = \frac{7 \pm \sqrt{22}}{3}.$$

  We can use the second derivative test to decide which is what:

  $$f''(x) = 6x - 14$$

  When $x = \frac{7+\sqrt{22}}{3}$, $f''(x) > 0$, so this is a local minimum. The second derivative is negative at the other critical point, so that is a local maximum.

- Find all inflections points of f.

  Solution: From the previous part, we see that $f''(7/3) = 0$. It is easy to see that the second derivative changes from negative to positive there, so that is the only inflection point.