# Math 125 Solutions to First Midterm

1. Compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

3 points (a) $\lim_{x\to 3} \frac{x^2-9}{6x(x-3)}$

**Solution:**

$$\lim_{x\to 3} \frac{(x-3)(x+3)}{6x(x-3)} = \lim_{x\to 3} \frac{(x+3)}{6x} = \frac{3+3}{18} = \frac{1}{3}.$$

3 points (b) $\lim_{x\to\infty} 6\cos\left(\frac{\pi}{x}\right)$

**Solution:**

$$\lim_{x\to\infty} 6\cos(\pi/x) = 6\cos(0) = 6.$$

3 points (c) $\lim_{x\to 2} \frac{x^2}{(x-2)^2}$

**Solution:** Note for $x$ close to 2, the numerator is close to 4 while the denominator tends towards zero. Thus, the function becomes unbounded at 2. Note also that the denominator is always positive. Hence, the limit is $+\infty$.

2. More of the same: compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

3 points (a) $\lim_{x\to\infty} \frac{x^2-1}{6x(x-1)}$

**Solution:** For $x$ very large, $x^2-1 \approx x^2$, and $x-1 \approx x$. Thus

$$\lim_{x\to\infty} \frac{x^2-1}{6x(x-1)} = \lim_{x\to\infty} \frac{x^2}{6x(x)} = \lim_{x\to\infty} \frac{1}{6} = \frac{1}{6}$$

3 points (b) $\lim_{h\to 3} \frac{(x+h)^2-x^2}{h}$

**Solution:**

$$\lim_{h\to 3} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 3} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 3} \frac{2xh+h^2}{h} = \lim_{h\to 3} 2x+h = 2x+3.$$

3 points (c) $\lim_{x\to-\infty} e^x \cos(x)$

**Solution:** Observe that for any $x$, we have $-1 \leq \cos(x) \leq 1$, and so we also have $-e^x \leq e^x \cos(x) \leq e^x$. Applying the squeeze theorem,

$$\lim_{x\to-\infty} (-e^x) \leq \lim_{x\to-\infty} e^x \cos(x) \leq \lim_{x\to-\infty} (e^x),$$

that is,

$$0 \leq \lim_{x\to-\infty} e^x \cos(x) \leq 0.$$

Hence, the limit is 0.

3. Let $f(x) = 2x^3 - 5x + 2$.

3 points (a) Find the slope of the secant line passing through the points on the curve $y = f(x)$ where $x = 0$ and $x = 1$.

**Solution:** The slope of a line is the ratio of the change in $y$ to the change in $x$. Here we have

$$\text{slope} = \frac{f(1) - f(0)}{1 - 0} = \frac{-1-2}{1} = -3.$$

3 points (b) Find $f'(1)$.

**Solution:** Using the power rule, $f'(x) = 6x^2 - 5$, so $f'(1) = 1$.

3 points (c) Write the equation of the tangent line to the graph of $y = f(x)$ when $x = 1$.

**Solution:** The point $(1, f(1))$ is on both the curve and the line. Now, $f(1) = 2 - 5 = -3$. We just need the equation of the line of slope 1 passing through the point $(1, -3)$. This is

$$y + 3 = (x - 1) \qquad \text{or} \qquad y = x - 4.$$

3 points (d) At $x = 1$, is $f(x)$ concave up, concave down, or neither? Justify your answer fully.

**Solution:** Since $f''(x) = 12x$, we know $f''(1) > 0$. Thus $f(x)$ is concave up at $x = 1$.

8 points 4. For what values of $x$ is the function $f(x) = \dfrac{e^x}{4 - e^{1/x}}$ continuous?

**Solution:** Since $f(x)$ is a composition of exponentials and rational functions, it is continuous everywhere on its domain.

Since $1/x$ is not defined for $x = 0$, the function is not continuous there.

Furthermore, there will be a discontinuity when the denominator is zero. That is, where $4 - e^{1/x} = 0$, or

$$\begin{aligned} 4 &= e^{1/x} \\ \ln(4) &= \ln\left(e^{1/x}\right) = 1/x \\ x &= \frac{1}{\ln(4)}. \end{aligned}$$

Thus, $f(x)$ is continuous at all real numbers except $x = 0$ and $x = \frac{1}{\ln(4)}$.

8 points 5. Write a limit that represents the slope of the graph

$$y = \begin{cases} |x|^x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

at $x = 0$. You **do not need to evaluate the limit.**

**Solution:** We just use the definition of the derivative at $x = 0$:

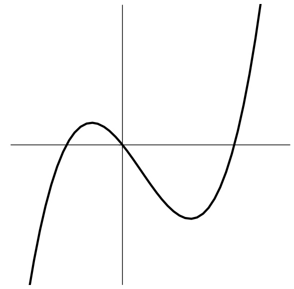
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}.$$

Since $h$ is not zero, $f(h) = |h|^h$ and $f(0) = 1$. So,

$$f'(0) = \lim_{h \to 0} \frac{|h|^h - 1}{h}.$$

If you prefer to use the version of the definition $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$, you get the same answer except with $x$ instead of $h$.

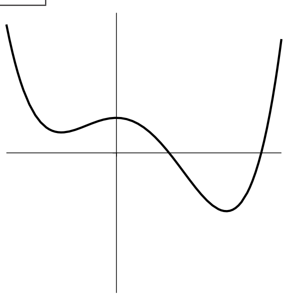
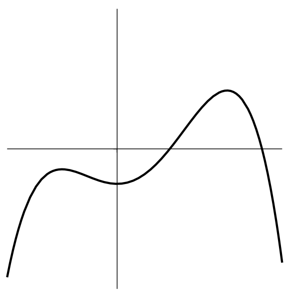
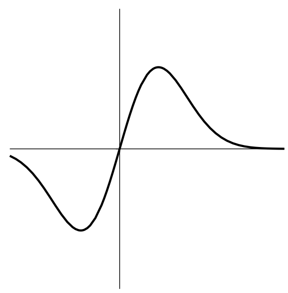
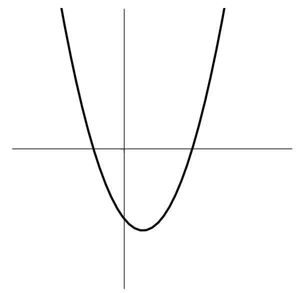
6. At right is the graph of **the derivative** $f'(x)$ of a function $f(x)$. Use it to answer each of the following questions.

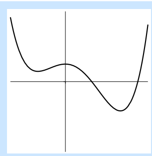


4 points (a) Is $f(x)$ concave up, concave down, or neither at $x = 0$?

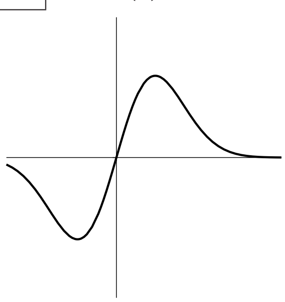
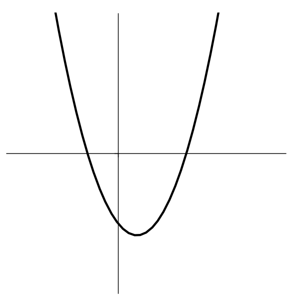
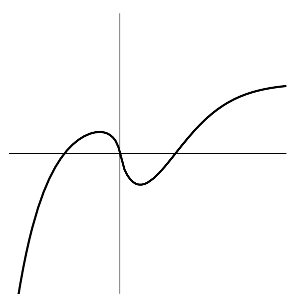
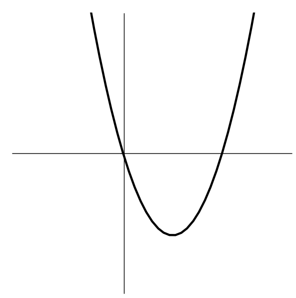
**Solution:** Since the derivative is decreasing at $x = 0$, we know $f(x)$ is concave down there.

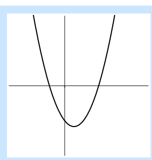
4 points (b) Which of the following best represents the graph of $f(x)$? (circle your answer).

**Solution:** The graph of $f(x)$ is 

4 points (c) Which of the following best represents the graph of $f''(x)$? (circle your answer).

**Solution:** The graph of $f''(x)$ is 

7. Let $f(x) = \dfrac{x^2 - 4x}{2(x^2 - 16)}$

4 points (a) Identify the horizontal asymptotes of $f(x)$. If there are none, write "NONE".

**Solution:** A function $f(x)$ has a horizontal asymptote at $y = L$ when $\lim_{x\to\infty} f(x) = L$. So, we have

$$\lim_{x\to\infty} \frac{x^2 - 4x}{2(x^2 - 16)} = \lim_{x\to\infty} \frac{x^2}{2x^2} = \frac{1}{2}.$$

Thus, there is a horizontal asymptote $y = \frac{1}{2}$.

4 points (b) Identify the vertical asymptotes of $f(x)$. If there are none, write "NONE".

**Solution:** We have a vertical asymptote at $x = a$ whenever $\lim_{x\to a^\pm} f(x) = \pm\infty$. The denominator of $f(x)$ factors as $2(x-4)(x+4)$, so we have to look at $a = 4$ and $a = -4$. Note that if $x \neq 4$ $x \neq -4$, we have

$$f(x) = \frac{x^2 - 4x}{2(x^2 - 16)} = \frac{x(x-4}{2(x-4)(x+4)} = \frac{x}{2(x+4)}$$

Near $x = -4$, we have

$$\lim_{x\to -4^+} f(x) = +\infty \qquad \text{and} \qquad \lim_{x\to -4^-} f(x) = -\infty,$$

so there is a vertical asymptote at $x = -4$.

Near $x = 4$, we have

$$\lim_{x\to 4} f(x) = \lim_{x\to 4} \frac{x}{2(x+4)} = \frac{4}{2(4+4)} = \frac{1}{4}.$$

Thus, $x = 4$ is not a vertical asymptote.

8 points 8. Write a function which expresses the area of a rectangle with a perimeter of 16 feet in terms of its width.

**Solution:** Let's let $W$ denote the width of the rectangle (in feet), and $L$ denote its length. Since the perimeter is 16, we know that

$$2L + 2W = 16,$$

or equivalently, $L = 8 - W$.

Since the area of the rectangle is $LW$, we have

$$A(W) = (8 - W)W$$