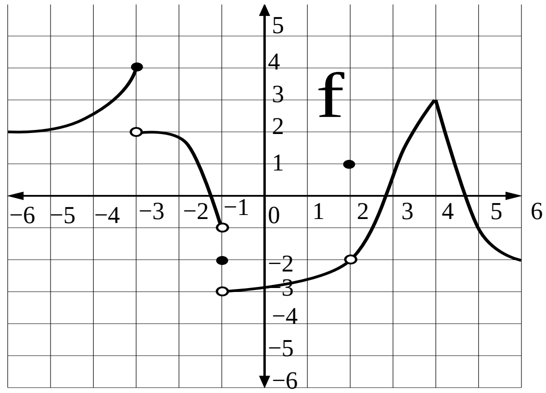
MATH 125 Solutions to First Midterm

1. The graph of a function f is shown below.



   (a) $\boxed{\text{3 points}}$ List all points $-6 \le x \le 6$ where $f(x)$ is not continuous. If there are none, write "none".

   Solution: $f(x)$ is not continuous at $x = -3$, since the graph has a "jump" there. More precisely, $\lim_{x\to-3} f(x)$ does not exist, so there is no way it can equal $f(-3)$. Similarly, $f(x)$ is not continuous at $x = -1$ for the same reasons. Finally, it is not continuous at $x = 2$, since $\lim_{x\to 2} f(x) = -2$, but $f(2) = 1$.

   So, the answer is $f(x)$ is not continuous at -3, -1, and 2.

   (b) $\boxed{\text{3 points}}$ What is $\lim\limits_{x\to-1^+} f(x)$? If it does not exist, write DNE.

   Solution: $\lim\limits_{x\to-1^+} f(x) = -3$: as x heads towards -1, the height of the graph gets close to -3. (The fact that $f(-1) = -2$ is irrelevant.)

   (c) $\boxed{\text{3 points}}$ Is $f(x)$ continuous from the left at $x = -3$?

   Solution: Yes, since $\lim\limits_{x\to-3^-} f(x) = 4 = f(-3)$.

   (d) $\boxed{\text{3 points}}$ What is $\lim\limits_{x\to 4}(f(x/2)+f(x+1))$

   Solution: We can compute the two parts of the sum separately. That is,

$$\lim_{x\to 4}(f(x/2)+f(x+1)) = \left(\lim_{x\to 4} f(x/2)\right) + \left(\lim_{x\to 4} f(x+1)\right)$$

   For the first term, notice that as $x \to 4$, $x/2 \to 2$, so

$$\lim_{x\to 4} f(x/2) = \lim_{z\to 2} f(z) = -2.$$

   For the second, we have $\lim\limits_{x\to 4} f(x+1) = \lim\limits_{w\to 5} f(w) = f(5) = -1$.

   So the answer is $-2+(-1) = -3$.

2. Let $h(x) = \sqrt{\dfrac{x+1}{x}}$.

   (a) $\boxed{\text{3 points}}$ What is the domain of $h(x)$?

   Solution: We must determine for which x the function makes sense. First, notice that we can rewrite $h(x)$ as $h(x) = \sqrt{1+\dfrac{1}{x}}$; this makes it clear that $h(x)$ will be defined only when both $x \neq 0$ and when $1+\frac{1}{x} \geq 0$.
   This second condition holds when $x \leq -1$ or $x > 0$. This implies $x \neq 0$, so the domain of $h(x)$ is $x \leq -1$ or $x > 0$.

   (b) $\boxed{\text{3 points}}$ Find two functions f and g so that $h = f \circ g$.

   Solution: There are many correct choices here. The most obvious (to me) is

   $$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \frac{x+1}{x}.$$

   (c) $\boxed{\text{4 points}}$ Write a formula for $h^{-1}(x)$.

   Solution: We write $y = h(x)$ and solve for x to get $h^{-1}(y)$. So:

   $$\begin{aligned} y &= \sqrt{\frac{x+1}{x}} \\ y^2 &= \frac{x+1}{x} \\ xy^2 &= x+1 \\ xy^2 - x &= 1 \\ x(y^2-1) &= 1 \\ x &= \frac{1}{y^2-1} = h^{-1}(y) \end{aligned}$$

   Thus,

   $$h^{-1}(x) = \frac{1}{x^2-1}.$$

3. (a) $\boxed{\text{3 points}}$ If $5e^{2x} = 10$, what is x?

   Solution: First, divide both sides by 5 to get $e^{2x} = 2$. Then take the natural log of both sides, to get

   $$2x = \ln 2 \qquad \text{so} \qquad x = \frac{\ln 2}{2}$$

(b) $\boxed{\text{3 points}}$ Solve $\ln(x^2) = 8$ for x. If there are no solutions, write "none".

Solution: Exponentiating both sides gives $x^2 = e^8$, so

$$x = \pm\sqrt{e^8} \qquad \text{that is,} \qquad x = \pm e^4.$$

(c) $\boxed{\text{3 points}}$ What is the inverse of the function $f(x) = x^2$, with $x < 0$? If the function has no inverse, write "no inverse".

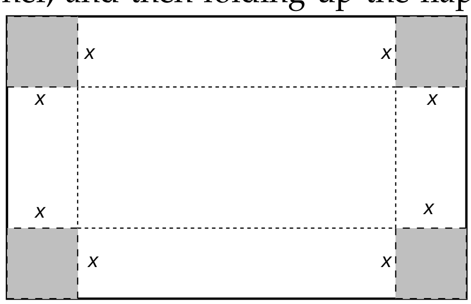
Solution: Most people got this one wrong. Let me rephrase the question:

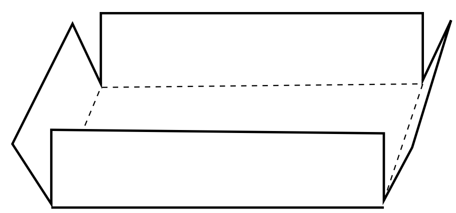
Suppose $y = x^2$, and x is negative. Write x in terms of y.

Since $y = x^2$, we know $x = \pm\sqrt{y}$. Do we want the $+$ or the $-$? Since x is negative, we obviously want the $-$. So the answer is

$$f^{-1}(x) = -\sqrt{x}.$$

4. A box without a top is to be made from a rectangular piece of cardboard which is 12 inches by 20 inches by cutting out four equal squares of side length x inches from each corner, and then folding up the flaps to form the sides of the box (see figure).





(a) $\boxed{\text{6 points}}$ Express the volume of the box V as a function of x.

Solution: The volume of the box is given by $V = (length)(width)(height)$.

The height of the box will be x, since that is how long the flap we fold up is.

The width of the box is $20 - 2x$, because we started with a piece of cardboard 20" wide, and cut x" off either end.

Similarly, the length will be $12 - 2x$.

Multiplying them all together gives

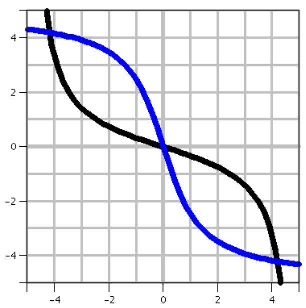
$$V(x) = x(20 - 2x)(12 - 2x)$$

(b) $\boxed{\text{3 points}}$ What is the domain of the function $V(x)$?

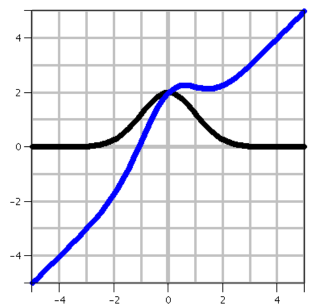
Solution: Remember, the domain is the values of x that are valid. We can't cut less than nothing off the piece of cardboard, so $x \geq 0$. Similarly, we can't cut more than half the smaller dimension, so $x \leq 6$. This means the domain is

$$0 \leq x \leq 6.$$

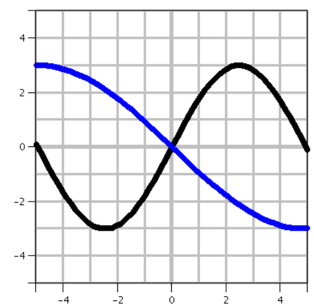
5. $\boxed{\text{9 points}}$ The graphs of several functions $f(x)$ are shown below. On the same set of axes, sketch the graph of the function $g(x)$ as indicated.



$g(x) = f^{-1}(x)$



$g(x) = f(x) + x$



$g(x) = -f(x/2)$

Solution: For the first graph, remember that if $y = f(x)$, then $f^{-1}(y) = x$. So the graph of $f^{-1}(x)$ is obtained from that of $f(x)$ by exchanging the roles of x and y. That is, we reflect the graph through the line $y = x$.

For the center graph, we want to add x to $f(x)$. This means that when $x < 0$, the graph will be shifted down, and when $x > 0$, the graph shifts up. For example, at the left edge, the answer should be at $f(-5) - 5$, which is just about -5. In the middle, the answer is $f(0) + 0 = f(0)$, and at the right edge, it is $f(5) + 5$.

The last graph is $-f(x/2)$, which means we first stretch the graph horizontally by a factor of 2 to get $f(x/2)$, then flip through the x-axis. If you are confused, think what value should go over, say, $x = 4$: we want $-f(4/2)$ which is $-f(2)$. Since $f(2) \approx 3$, the solution should go near $(4, -3)$. If you do this for several other points, you'll see why the solution is what it is.

6. Compute each of the following limits. If the limit is undefined, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE).

(a) $\boxed{\text{3 points}}$ $\lim\limits_{x \to 2} xe^{x-2}$

Solution: Since xe^{x-2} is continuous for all x, we just plug in to get

$$\lim_{x \to 2} xe^{x-2} = 2e^0 = 2 \cdot 1 = 2.$$

(b) $\boxed{\text{3 points}}$ $\displaystyle\lim_{h\to 0} \frac{(3+h)^2-9}{h}$

Solution: Attempting to plug in $h=0$ gives us the indeterminate form $\frac{0}{0}$, so we need to do a little algebra:

$$\lim_{h\to 0} \frac{(3+h)^2-9}{h} = \lim_{h\to 0} \frac{(9+6h+h^2)-9}{h} = \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0} 6+h = 6.$$

(c) $\boxed{\text{3 points}}$ $\displaystyle\lim_{x\to +\infty} \frac{3x^2-22x+7}{x^2-49}$

Solution: Since $x \to +\infty$, we divide top and bottom by the highest power of x to get

$$\lim_{x\to +\infty} \frac{\frac{3x^2}{x^2} - \frac{22x}{x^2} + \frac{7}{x^2}}{\frac{x^2}{x^2} - \frac{49}{x^2}} = \lim_{x\to +\infty} \frac{3 - \frac{22}{x} + \frac{7}{x^2}}{1 - \frac{49}{x^2}} = \frac{3-0+0}{1-0} = 3$$

Of course, this can also be done more efficiently by neglecting all but the fastest-growing terms on the top and bottom (this is really the same thing):

$$\lim_{x\to +\infty} \frac{3x^2-22x+7}{x^2-49} = \lim_{x\to +\infty} \frac{3x^2}{x^2} = \lim_{x\to +\infty} \frac{3}{1} = 3.$$

(d) $\boxed{\text{3 points}}$ $\displaystyle\lim_{x\to +\infty} \cos\left(\frac{\pi}{x}\right)$

Solution: Since the cosine is a continuous function, we have

$$\lim_{x\to +\infty} \cos\left(\frac{\pi}{x}\right) = \cos\left(\lim_{x\to +\infty} \frac{\pi}{x}\right) = \cos(0) = 1.$$

(e) $\boxed{\text{3 points}}$ $\displaystyle\lim_{x\to 4} \frac{2+\sqrt{x}}{2-\sqrt{x}}$

Solution: As $x \to 4$, the function tends to $\frac{4}{0}$. No amount of algebra will change the fact that we are dividing a nonzero number by something tending to zero, so the limit will either be $+\infty$, $-\infty$, or DNE. We need to determine which it is.

Notice that if $x < 4$, the denominator is positive, so $\displaystyle\lim_{x\to 4^-} \frac{2+\sqrt{x}}{2-\sqrt{x}} = -\infty$.

On the other hand $x > 4$, the denominator is negative, so $\displaystyle\lim_{x\to 4^+} \frac{2+\sqrt{x}}{2-\sqrt{x}} = +\infty$.

Since the one-sided limits are different, the two-sided limit does not exist (DNE).

7. Let $q(x) = \begin{cases} \frac{x-1}{x+3} & x < 1 \\ x+3 & x \geq 1 \end{cases}$

(a) $\boxed{\text{3 points}}$ Calculate $\lim_{x\to 1^-} q(x)$. If the limit does not exist, write DNE.

Solution: Since we want $x \to 1^-$, we are only considering $x < 1$, so $q(x) = \frac{x-1}{x+3}$. Thus we have

$$\lim_{x\to 1^-} q(x) = \lim_{x\to 1^-} \frac{x-1}{x+3} = \frac{1-1}{1+3} = 0.$$

(b) $\boxed{\text{3 points}}$ Calculate $\lim_{x\to 1^+} q(x)$. If the limit does not exist, write DNE.

Solution: This time we are considering $x > 1$, so we have

$$\lim_{x\to 1^+} q(x) = \lim_{x\to 1^+} x+3 = 1+3 = 4.$$

(c) $\boxed{\text{3 points}}$ For what x is $q(x)$ continuous?

Solution: From the previous two parts, we know $q(x)$ can't be continuous at $x = 1$. But notice that when $x < 1$, $q(x) = \frac{x-1}{x+3}$, which is not defined when $x = -3$.

Everywhere else, $q(x)$ is continuous and well-defined, so $q(x)$ is continuous for all x except $x = 1$ and $x = -3$.

8. $\boxed{\text{8 points}}$ The equation $1 + \sin\left(\frac{\pi}{4}x^2\right) - 3x = 0$ has exactly one solution for $0 \leq x \leq 5$. Between what two (closest) whole numbers does the solution lie? You must **fully** justify your answer to receive credit.

Solution: Since $f(x) = 1 + \sin\left(\frac{\pi}{4}x^2\right) - 3x$ is a continuous function, we can apply the Intermediate Value Theorem. We want to find two integers x_1 and x_2 that differ by one and so that $f(x_1) > 0$ but $f(x_2) < 0$. So we just try some.

$$f(0) = 1 + \sin(0) - 3 \cdot 0 = 1 > 0.$$

$$f(1) = 1 + \sin\left(\frac{\pi}{4}\right) - 3 \cdot 1 = -2 + \frac{\sqrt{2}}{2} < 0.$$

Since we just found that $f(0) > 0$ and $f(1) < 0$, we can stop. The solution must lie between 0 and 1.