Homework 3. Due Nov. 7
Math 254a. Topics in Real Analysis, Fall 2007

Note: For the questions below, it may be useful to use Jones' theorem about Lipschitz and BiLipschitz functions for Lipschitz functions

$$f : [0,1]^k \to \mathbb{R}^n.$$

(you may use it with $1 \leq k$.)

1. Let $f : [0,1]^k \to \mathbb{R}^n$ be Lipschitz. Let $E = f([0,1]^k)$. Show that for $H^k - a.e.\ x \in E$ we have $f'(x)$ has rank k.

2. Let E, f be as above. Show that for $H^k - a.e.\ x \in E$ we have

   $$\theta^k_*(E, x) > 0\,.$$

3. $\gamma : [0,1]^k \to \mathbb{R}^n$ is called a *k-Ahlfors-David-regular map* with constant C iff:
   γ is C Lipschitz and $H^k(\gamma^{-1}\text{ball}(x,r)) \leq Cr^k$

   Show that there is an $\alpha > 0$ such that if γ is as above, then for any cube $Q \subset [0,1]$ there is a k-plane P in $\mathbb{R}^n$ (not necessarily going through the origin) such that $H^k(\pi_P \circ \gamma(Q)) \geq \alpha H^k(Q)$. ($\pi_P$ here is the orthogonal projection of $\mathbb{R}^n$ onto P).

   Hint: use a normal family argument (like we did in the proof of Jones' theorem about Lipschitz and BiLipschitz functions) and then use question (1).