MAT 142 Spring 2003
FINAL

!!! WRITE YOUR NAME, STUDENT ID BELOW !!!

NAME : ID :

THERE ARE 11 PROBLEMS. THEY DO NOT HAVE EQUAL VALUE.
SHOW YOUR WORK!!!

1	40	
2	30	
3	30	
4	20	
5	30	
6	30	
7	40	
8	dropped	
9	30	
10	20	
11	30	
Total	300	

1. Solve the following differential equation with initial value:

$$\frac{dy}{dx} = x + y, \qquad y(0) = 0.$$

$y' - y = x$

$y' + Py = Q$.

$v = e^{\int P} = e^{-x}$

$y'e^{-x} - ye^{-x} = xe^{-x}$

$(ye^{-x})' = xe^{-x}$

$ye^{-x} = \int xe^{-x}dx$

$= -xe^{-x} + \int e^{-x}dx$

$= -xe^{-x} - e^{-x} + C$

$y = -x - 1 + Ce^{x}$

$0 = -1 + C \Rightarrow C = 1$

$y = e^{x} - x - 1$

Check: $y' = e^{x} - 1 = (e^{x} - x - 1) + x$ ✓

$y(0) = 1 - 0 - 1 = 0$ ✓

2. A semicircular plate $2\,ft$ in diameter is submerged vertically into a fluid with the diameter along the surface. The weight-density of the fluid is $60\,lb/ft^3$.

Find the force exerted by the fluid on one side of the plate.

Not Covered in Class

3. Use the definition of the natural logarithm

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0,$$

to prove that, for $a > 0$:

$$\ln ax = \ln a + \ln x.$$

$$\frac{d}{dx} \ln ax = \frac{d}{dx}\int_1^{ax} \frac{dt}{t}, \qquad u = ax, \quad \frac{du}{dx} = a$$

$$y = \ln ax = \int_1^{ax} \frac{dt}{t}.$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}. \qquad \frac{dy}{du} = \frac{d}{du}\int_1^{u} \frac{dt}{t} = \frac{1}{u} = \frac{1}{ax}.$$

$$\frac{dy}{dx} = \frac{1}{ax} \cdot a = \frac{1}{x}.$$

Also! $\dfrac{d}{dx}\displaystyle\int_1^{x} \frac{dt}{t} = \frac{1}{x}$. Therefore,

$\ln ax - \ln x = C$. Plugging in $x=1 \Rightarrow$

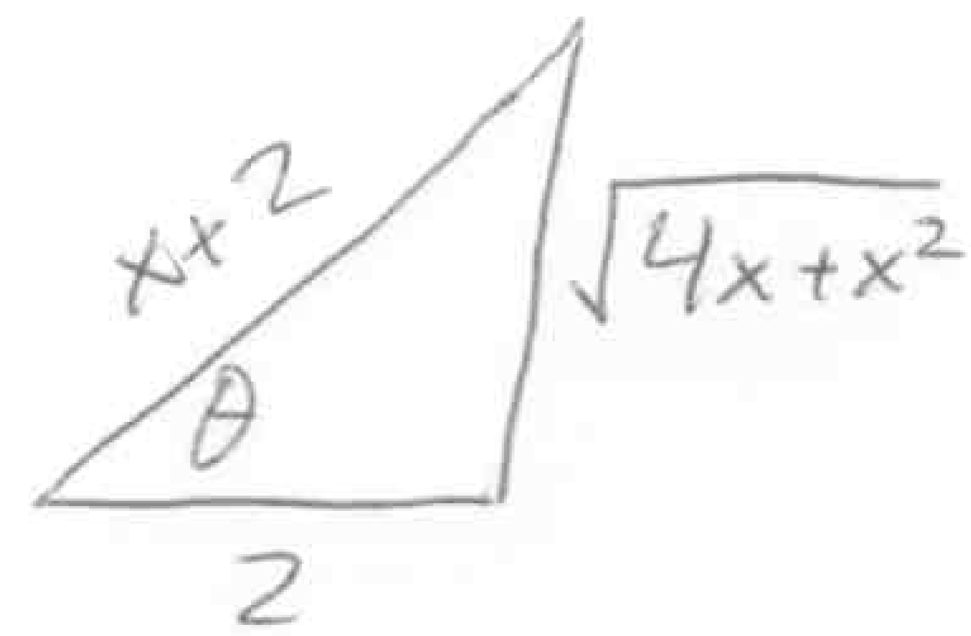
$\ln a - \underbrace{\cancel{\ln 1}}_{0} = C \Rightarrow C = \ln a.$

$\therefore \ln ax = \ln a + \ln x.$

4. Integrate:

a)

$$\int \frac{1}{\sqrt{4x+x^2}}\,dx.$$



$$x^2+4x = (x+2)^2-4.$$

$$2\sec\theta = x+2$$
$$2\sec\theta\tan\theta\,d\theta = dx$$
$$2\tan\theta = \sqrt{4x+x^2}$$

$$\int \frac{2\sec\theta\tan\theta\,d\theta}{2\tan\theta} = \int \sec\theta\,\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\,d\theta$$
$$= \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta$$
$$= \ln|\sec\theta+\tan\theta| + C$$
$$= \ln\left|\frac{x+2}{2}+\frac{\sqrt{4x+x^2}}{2}\right| + C$$

b)

$$\int \cos(\ln t)\,dt.$$

$$u=\ln t,\quad du=\frac{dt}{t} \Leftrightarrow t\,du=dt \Leftrightarrow e^u\,du=dt$$
$$e^u=t$$

$$I=\int e^u\cos u\,du = e^u\sin u-\int e^u\sin u\,du$$
$$= e^u\sin u+e^u\cos u-\int e^u\cos u\,du$$
$$= e^u\sin u+e^u\cos u-I \Rightarrow$$
$$2I=e^u\sin u+e^u\cos u+C \Rightarrow$$
$$I=\frac{e^u}{2}(\sin u+\cos u)+C'.$$

5.

a) Find the Taylor series generated by $f(x) = x^5$ at $x = 1$.

$f'(x) = 5x^4$, $f''(x) = 5 \cdot 4x^3$, $f'''(x) = 5 \cdot 4 \cdot 3x^2$, $f^{(4)}(x) = 5 \cdot 4 \cdot 3 \cdot 2x$

$f^{(5)}(x) = 5!$, $f^{(n)}(x) = 0$, $n \geq 6$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = 1 + 5(x-1) + 10(x-1)^2 + 10(x-1)^3 + 5(x-1)^4 + (x-1)^5$$

(or use $((x-1)+1)^5 = \sum_{k=0}^{5} \binom{5}{k}(x-1)^k$).

b) Use a), or any other method, to write the following expression using partial fractions:

$$y = \frac{x^5}{(x-1)^6}.$$

$$y = \frac{1}{(x-1)^6} + \frac{5}{(x-1)^5} + \frac{10}{(x-1)^4} + \frac{10}{(x-1)^3} + \frac{5}{(x-1)^2} + \frac{1}{x-1}$$

6. Find a value c that makes the function $f(x)$ continuous at $x = 0$, where

$$f(x) = \frac{\sin(-7x)}{\tan(11x)}, \quad x \neq 0, \qquad f(0) = c.$$

$$\lim_{x \to 0} \frac{\sin(-7x)}{\tan(11x)} = \lim_{x \to 0} \frac{-7\cos(-7x)}{11\sec^2(11x)} = -\frac{7}{11}.$$

$$c = -\frac{7}{11}$$

7. Study the convergence of the following series.

a)

$$\sum_{n=10}^{\infty} \frac{(\ln n)^2}{n^4}$$

$\ln n < n \quad \forall n \in \mathbb{N}.$

$\therefore \quad \frac{(\ln n)^2}{n^4} < \frac{n^2}{n^4} = \frac{1}{n^2}. \qquad \sum \frac{1}{n^2}$ conv $\Rightarrow \sum_{n=10}^{\infty} \frac{(\ln n)^2}{n^4}$

converges by the comparison test.

b)

$$\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$ Ratio test!

$\frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} = \frac{n+1}{(2n+2)(2n+1)}$. as $n \to \infty$,

the ratio goes to 0.

$\therefore$ Converges.

c)

$$\sum_{n=1}^{\infty}(-1)^{n+1}(1+\frac{1}{n})$$

terms do not go to 0.

diverges.

8. Find a sequence a_n such that all three conditions below are satisfied:

1) $\lim_{n\to\infty} a_n = 0$,

2) $a_n \geq 0$ for all n and

3) $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ diverges.

$$a_n = \begin{cases} 0, & n \text{ odd} \\ \frac{n}{2}, & n \text{ even.} \end{cases}$$

$$\Rightarrow S_k = \sum_{n=1}^{k} (-1)^{n+1} a_n =$$

$$0 - 1 + 0 - \frac{1}{2} + 0 - \frac{1}{3} + 0 - \frac{1}{4} + \cdots$$

$$= -\sum_{n=1}^{k} \frac{1}{n} \quad \text{diverges.}$$

9. Study the following power series (radius of convergence, interval of convergence, convergence at the endpoints).

a)

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}.$$

ratio test: $\dfrac{2^{n+1}|x|^{n+1}}{(n+1)!} \cdot \dfrac{n!}{2^n |x|^n} = \dfrac{2|x|}{n+1} \to 0 \ \forall x \in \mathbb{R}$.

$\therefore$ Interval of convergence is $(-\infty, \infty)$,

radius of conv. $= \infty$.

(or: observe $= e^{2x}$)

b)

$$\sum_{n=2}^{\infty} \frac{(x-1)^n}{\sqrt{n}\ln n}.$$

ratio: $\dfrac{|x-1|^{n+1}}{\sqrt{n+1}\,\ln(n+1)} \cdot \dfrac{\sqrt{n}\,\ln n}{|x-1|^n} = \dfrac{\sqrt{n}}{\sqrt{n+1}}\,\dfrac{\ln n}{\ln(n+1)}\,|x-1$

$\dfrac{\sqrt{n}}{\sqrt{n+1}} = \dfrac{1}{\sqrt{1+\frac{1}{n}}} \to 1.$ $\lim\limits_{\cancel{y}\to\infty} \dfrac{\ln \cancel{y}}{\ln(\cancel{y}+1)} = \lim\limits_{y\to\infty} \dfrac{\frac{1}{y}}{\frac{1}{y+1}}$

$= \lim\limits_{y\to\infty} \dfrac{y+1}{y} = 1.$ $\therefore$ ratio $\to |x-1|.$

radius of conv. $= 1$, $x = 0$: $\sum \dfrac{(-1)^n}{\sqrt{n}\ln n}$ conv.

by alt. ser. test $x = 2$: $\sum \dfrac{1}{\sqrt{n}\ln n}$?

$\dfrac{1}{\sqrt{n}\ln n} \geq \dfrac{1}{n \ln n}.$ Claim: $\sum \frac{1}{n\ln n}$ diverges.

pf: $\displaystyle\int_2^{b} \frac{dx}{x\ln x}$ $\quad u = \ln x$, $du = \dfrac{dx}{x}$.

$\displaystyle\int_{\ln 2}^{\ln b} u^{-1}\,du = \ln u \Big|_{\ln 2}^{\ln b} = \ln(\ln b) - \ln(\ln 2).$

Taking the limit as $b \to \infty$, the $\int$ diverges,

so the sum diverges, and the other sum diverges by the comp. test.

$\therefore$ Interval of conv $= [0, 2).$

10. Find the Maclaurin series of the following function

$$f(x) = x \ln(1 + x^2).$$

$$\ln(1+x) = \int_1^x \frac{dt}{1+t} = \int_1^x \sum_{n=0}^{\infty} (-1)^n t^n =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

$$x \ln(1+x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n+1}}{n}$$

11. Prove, using (ϵ, N), that

$$\lim_{n\to\infty}(-1 - \frac{1}{n^{\frac{2}{3}}}) = -1.$$

Not required (not done in class)