

**The 67th William Lowell Putnam Mathematical Competition**  
**Saturday, December 2, 2006**

- A1 Find the volume of the region of points  $(x, y, z)$  such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

- A2 Alice and Bob play a game in which they take turns removing stones from a heap that initially has  $n$  stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many  $n$  such that Bob has a winning strategy. (For example, if  $n = 17$ , then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

- A3 Let  $1, 2, 3, \dots, 2005, 2006, 2007, 2009, 2012, 2016, \dots$  be a sequence defined by  $x_k = k$  for  $k = 1, 2, \dots, 2006$  and  $x_{k+1} = x_k + x_{k-2005}$  for  $k \geq 2006$ . Show that the sequence has 2005 consecutive terms each divisible by 2006.

- A4 Let  $S = \{1, 2, \dots, n\}$  for some integer  $n > 1$ . Say a permutation  $\pi$  of  $S$  has a local maximum at  $k \in S$  if

- (i)  $\pi(k) > \pi(k+1)$  for  $k = 1$ ;
- (ii)  $\pi(k-1) < \pi(k)$  and  $\pi(k) > \pi(k+1)$  for  $1 < k < n$ ;
- (iii)  $\pi(k-1) < \pi(k)$  for  $k = n$ .

(For example, if  $n = 5$  and  $\pi$  takes values at 1, 2, 3, 4, 5 of 2, 1, 4, 5, 3, then  $\pi$  has a local maximum of 2 at  $k = 1$ , and a local maximum of 5 at  $k = 4$ .) What is the average number of local maxima of a permutation of  $S$ , averaging over all permutations of  $S$ ?

- A5 Let  $n$  be a positive odd integer and let  $\theta$  be a real number such that  $\theta/\pi$  is irrational. Set  $a_k = \tan(\theta + k\pi/n)$ ,  $k = 1, 2, \dots, n$ . Prove that

$$\frac{a_1 + a_2 + \dots + a_n}{a_1 a_2 \dots a_n}$$

is an integer, and determine its value.

- A6 Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.

- B1 Show that the curve  $x^3 + 3xy + y^3 = 1$  contains only one set of three distinct points,  $A, B$ , and  $C$ , which are vertices of an equilateral triangle, and find its area.

- B2 Prove that, for every set  $X = \{x_1, x_2, \dots, x_n\}$  of  $n$  real numbers, there exists a non-empty subset  $S$  of  $X$  and an integer  $m$  such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

- B3 Let  $S$  be a finite set of points in the plane. A linear partition of  $S$  is an unordered pair  $\{A, B\}$  of subsets of  $S$  such that  $A \cup B = S$ ,  $A \cap B = \emptyset$ , and  $A$  and  $B$  lie on opposite sides of some straight line disjoint from  $S$  ( $A$  or  $B$  may be empty). Let  $L_S$  be the number of linear partitions of  $S$ . For each positive integer  $n$ , find the maximum of  $L_S$  over all sets  $S$  of  $n$  points.

- B4 Let  $Z$  denote the set of points in  $\mathbb{R}^n$  whose coordinates are 0 or 1. (Thus  $Z$  has  $2^n$  elements, which are the vertices of a unit hypercube in  $\mathbb{R}^n$ .) Given a vector subspace  $V$  of  $\mathbb{R}^n$ , let  $Z(V)$  denote the number of members of  $Z$  that lie in  $V$ . Let  $k$  be given,  $0 \leq k \leq n$ . Find the maximum, over all vector subspaces  $V \subseteq \mathbb{R}^n$  of dimension  $k$ , of the number of points in  $V \cap Z$ . [Editorial note: the proposers probably intended to write  $Z(V)$  for “the number of points in  $V \cap Z$ ”, but this changes nothing.]

- B5 For each continuous function  $f : [0, 1] \rightarrow \mathbb{R}$ , let  $I(f) = \int_0^1 x^2 f(x) dx$  and  $J(f) = \int_0^1 x (f(x))^2 dx$ . Find the maximum value of  $I(f) - J(f)$  over all such functions  $f$ .

- B6 Let  $k$  be an integer greater than 1. Suppose  $a_0 > 0$ , and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for  $n > 0$ . Evaluate

$$\lim_{n \rightarrow \infty} \frac{a_n^{k+1}}{n^k}.$$