Math 200 Lecture 2 (Tu/Th) Exam II Fall 2008 Mark de Cataldo

Name and ID#: ____________________, _______________

Instructions:

1. Your work will be carefully graded. The correct answers without sufficient work, or not using the method required, will receive minimal or no credit. If you use a theorem from the book, you need to tell us which one you are using (give its name or its statement).

2. The point value of each problem occurs to the left of the problem.

3. Provide clearly written answers in the space provided. You can use the flip sides of the sheets as scratch paper. **Do not tear off any page. You must return all pages.**

4. No books, no cell phones, no PDAs, no calculators.

5. Use correct notation.

Problem	**Points Possible**	**Points**
1	20+30 =50	
2	30	
3	40	
4	40	
5	30	
6	15×4=60	
Total	250	

1. (50 total pts)

(a) (20 pts) State the pigeonhole principle for finite sets.

If X and Y are finite and non empty, with $|X| < |Y|$, then there is no injection $Y \to X$.

(b) (30 pts) Let X and Y be finite sets with $|X| < |Y|$. Prove, using the pidgeonhole principle, that there is no surjection $f : X \to Y$.

By contradiction, assume there is a surjection $f : X \to Y$.

For every $y \in Y$, let $x_y \in X$ be such that $f(x_y) = y$. x_y exists since f is surjective.

Let $g : Y \to X$ be defined by $g(y) = x_y$.

g is injective: if g were not injective, there would be $y_1 \neq y_2$ with $x_{y_1} = x_{y_2}$, but then $y_1 = f(x_{y_1}) = f(x_{y_2}) = y_2$, a contradiction.

The injectivity of g contradicts the pigeonhole principle. This contradiction shows f cannot exist.

2. (30 pts) Prove that $\sqrt{30}$ is not rational.

By contradiction, assume $\sqrt{30} = p/q$, in lowest terms.

Then $30q^2 = p^2$.

Then p^2 is even.

p^2 is even implies $p = 2b$ is even (no need to prove this, however, if $p = 2a + 1$ is odd, then p^2 is odd).

We have $30q^2 = 4b^2$.

So we have $15q^2 = 2b^2$.

So $15q^2$ is even. Then q^2 is even: if not, then as above, $15q^2$ would be odd.

Then, as above, q is even and p/q is not in lowest terms. Contradiction.

3. (40 pts) There are 19 people out of which we want to make two indoor soccer teams of 10 and 9. Team A should have 7 starting players and 3 alternates, and team B should have 7 starting players and 2 alternates. In how many different ways can we do this?

We choose 10 people for the first team. This leaves 9, so there is nothing left to choose, in terms of forming the teams:

$$\binom{19}{10} \times \binom{9}{9} = \binom{19}{10}.$$

We are not done. In the first team we have to choose 3 people, in the second 2:

$$\binom{10}{3} \times \binom{9}{2}.$$

The answer is the product

$$\binom{19}{10} \times \binom{9}{9} \times \binom{10}{3} \times \binom{9}{2}.$$

No need to simplify (but if you want, drop $\binom{9}{9} = 1$.)

4. (40 total pts)

(a) (20 pts) For $k \in \mathbb{Z}^+$ and $k \geq 2$, let $T_k := \{n \in \mathbb{Z} \,|\, n = km, m \in \mathbb{Z}\}$. Let

$$T = \bigcap_{k=2}^{5} T_k.$$

Prove that T is denumerable.

T contains all the multiples of $2 \times 3 \times 4 \times 5 = 120$ and is thus infinite.

T is a subset of the denumerable set $\mathbb{Z}^+$ and is thus countable (proposition in the book).

T is countable and infinite, so it is denumerable.

Or, prove that T is given by the multiples of 60 and then prove that is denumerable.

(b) (20 pts) Prove that $\mathbb{R}^2 - T^2$ is uncountable.

(You can use (a) even if you could not prove it.)

First observe that $\mathbb{R}^2$ is uncountable: it is infinite and it contains the uncountable $\mathbb{R}$ (for example as the x-axis); if it were countable, it would be denumerable so that every subset would be countable, contradiction.

The difference set in question is infinite: it contains, for example the points $(0, 1/n)$.

If it were countable, then it would be denumerable.

If it were denumerable then $\mathbb{R}^2 = \mathbb{R}^2 - T^2 \coprod T^2$ would be the union of two countable sets and then it would be countable (proposition in the book).

This contradicts the fact that $\mathbb{R}^2$ is uncountable.

Hence the difference set is uncountable.

5. (30 total pts)

(a) (10 pts) State the protractor axiom, parts (1), (2), (3) and (4).
See geometry notes.

(b) (20 pts) Let A,B,C,D be distinct points such that C and D lie on the same side of the line $\overleftrightarrow{AB}$. Prove that if $\overrightarrow{AD}$ is inside the angle $\angle BAC$, then $m(\angle BAD) < m(\angle BAC)$.

By the protractor axiom (4),

$$m(\angle BAC) = m(\angle BAD) + m(\angle DAC).$$

By the same axiom (1)

$$m(\angle DAC) > 0.$$

Hence

$$m(\angle BAD) < m(\angle BAC).$$

6. (60 total pts) Let Π be a set with n elements, $n \geq 5$ a fixed integer. Define the plane, as in the geometry notes, to be Π. We define the lines to be pairs of distinct points of Π.

(a) (15 pts) How many lines are there in Π?

As many as the subsets with two elements: $\binom{n}{2}$

(b) (15 pts) Is the incidence axiom true or false in this situation?

1. Yes: for any two points there is a unique line through them.
2. Yes: every line contains at least two distinct points.
3. Yes: in fact, for every line, there are $n - 2$ points not on that line.

The incidence axiom is true.

(c) (15 pts) Is the parallel axiom true or false in this situation?

It is false: let A,B,C,D,E be five distinct points. Take the line $\overleftrightarrow{AB}$ and the point C. We have the two distinct lines $\overleftrightarrow{CD}$ and $\overleftrightarrow{CE}$ which pass through C and are parallel to $\overleftrightarrow{AB}$ since they do not meet it.

(d) (15 pts) Is the ruler axiom true or false in this situation?

False. Any line has exactly two points and there cannot be a bijection between a line and the inifinte set of real numbers. (This is regardless of how we define distances.)