Math 200 Lecture 1 (Mo/Wed) Exam I Fall 2008 Scott Simon

1. (50 total pts)

(a) (25 pts) Write the contrapositive of the following sentence:

If Mary does not take the car, then John cannot jog to work.

P = Mary does not take the car;

Q = John cannot jog to work.

$P \Rightarrow Q \xrightarrow[\text{positive}]{\text{contra}} \sim Q \Rightarrow \sim P$

If John can jog to work, then Mary does take the car.

(b) (25 pts) Negate the following statement:

There exists a real number r such that, for every integer m and for every integer n, we have

$$nm < r \quad \text{or} \quad nm \geq 0.$$

(An answer like "it is not true that ..." will be given 0 points).

$\exists r \in \mathbb{R}, \forall m, n \in \mathbb{Z} : (nm < r) \text{ or } (nm \geq 0).$

Negation; $\forall r \in \mathbb{R}, \exists m, n \in \mathbb{Z} : (nm \geq r) \text{ and } (nm < 0).$

2. (25 pts) Prove that if $n^3 + 1$ is odd, then n is even.

For using contradiction, we assume; $n = 2k+1$ for some k.

$\longrightarrow n^3+1 = (2k+1)^3+1 = 8k^3 + 6k(2k+1)+1+1$

$= 2\underbrace{(4k^3 + 3k(2k+1)+1)}_{q} = 2q.$

$\longrightarrow n^3+1$ is even, contradicting n^3+1 is odd. ▨

3. (30 pts) Use to induction to prove that $n! \geq 2^n$ for every $n \geq 4$

$n = 2$;

$2! = 2 < 2^2$ (!).

$n = 4$;

$4! = 24 \geq 2^4 = 16$ ✓.

We know that $(n+1) \geq 2$ ⊛. $\forall n \geq 2$.

And we also assumed that the statement holds for n. $\longrightarrow n! \geq 2^n$ ⊛. $\longrightarrow n!(n+1) = (n+1)! \geq 2^{n+1}$

▨

4. (30 pts) Prove that

$$(A \cup C) - B = (A - B) \cup (C - B).$$

$x\in A$	$x\in B$	$x\in C$	$x\in A\cup C$	$x\in (A\cup C)-B$	$x\in A-B$	$x\in C-B$	$x\in (A-B)\cup(C-B)$
1	1	1	1	0	0	0	0
1	1	0	1	0	0	0	0
1	0	1	1	1	1	1	1
1	0	0	1	1	1	0	1
0	1	1	1	0	0	0	0
0	1	0	0	0	0	0	0
0	0	1	1	1	0	1	1
0	0	0	0	0	0	0	0

The two columns coincide; $\iff$ $x\in (A\cup C)-B \iff x\in (A-B)\cup(C-B)$.

5. (40 total pts) Let $f : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$, $f(m,n) = mn$.

(a) (20pts) Is f injective (prove or disprove)?

(b) (20pts) Is f surjective (prove or disprove)?

(a) $f(1,2) = 2$ whilst; $f(2,1) = 2$.
Not injective.

(b) $\forall m \in \mathbb{Z}$; $f(m,1) = m$.
f is surjective.

6. (75 total pts) True or False (prove your answer or give a counterexample). In what follows, x, y and z are real numbers.

(a) (25pts) For every x, there is y such that $x^2 + y^2 < 0$.

False.

We know that $\forall x, y \in \mathbb{R};\ x^2 \geq 0\ ,\ y^2 \geq 0$

$\Rightarrow x^2 + y^2 \geq 0.$

(b) (25pts) There is x such that for every y, $x + y > 100$.

False.

For each x, we can consider y as ~~[illegible]~~

$y = -x \longrightarrow x + y = 0 < 100.$

(c) (25pts) There are x and y such that for every z, $-z^2 < xy$.

True.

We know that $-z^2 \leq 0$.

let $\begin{cases} x = +1 \\ y = +1 \end{cases} \longrightarrow z^2 \leq 0 < 1 = xy.$