Math 200 **Lecture 1 (Tu/Th)** **Exam II** **Fall 2008** **Scott Simon**

Name and ID#: ____________________, ______________

Instructions:

1. Your work will be carefully graded. The correct answers without sufficient work, or not using the method required, will receive minimal or no credit. If you use a theorem from the book, you need to tell us which one you are using (give its name or its statement).
2. The point value of each problem occurs to the left of the problem.
3. Provide clearly written answers in the space provided. You can use the flip sides of the sheets as scratch paper. **Do not tear off any page. You must return all pages.**
4. No books, no cell phones, no PDAs, no calculators.
5. Use correct notation.

Problem	**Points Possible**	**Points**
1	20+30 =50	
2	30	
3	30	
4	40	
5	40	
6	60	
Total	250	

1. (50 total pts)

(a) (20 pts) State the inclusion-exclusion principle for two finite sets.

Solution:
Let X, Y be finite sets. Then $|X \cup Y| = |X| + |Y| - |X \cap Y|$.

(b) (30 pts) Suppose that X is a non-empty finite set of cardinality n. Prove that a set Y has $|Y| = n$ if and only if there is a bijection $X \to Y$.

Solution:
By the definition of cardinality, there is a bijection $f : \mathbb{N}_n \to X$. Assume Y has cardinality n. Then there is a bijection $g : \mathbb{N}_n \to Y$. Therefore, $g \circ f^{-1} : X \to Y$ is a bijection. Conversely, if $h : X \to Y$ is a bijection, then so is $h \circ f : \mathbb{N}_n \to Y$, and hence $|Y| = n$.

2. (30 pts) Prove that $\sqrt{2^n}$ is rational if and only if n is even.

(If you use properties of $\sqrt{2}$, you must prove them.)

Solution:
If n is even, let $x = 2^{n/2} \in \mathbb{Z} \subseteq \mathbb{Q}$. Then $x^2 = (2^{n/2})^2 = 2^n$, and so $x = \sqrt{2^n}$. Now suppose $n = 2m + 1$ is odd and $\sqrt{2^n}$ is rational. Write $\sqrt{2^n} = p/q$, with p, q coprime positive integers. Thus $q^2 = 2^n p^2$. Therefore q is even, so p is odd. Write $q = 2^k a$, where a is odd. Then $2^{2k}a^2 = 2^{2m+1}p^2$. If $k \leq m$, then $a^2 = 2^{2(m-k)+1}p^2$, and so a is even, a contradiction. If $k > m$, then $2^{2(k-m)-1}a^2 = p^2$, and so p is even, a contradiction. Therefore, if n is odd, then $\sqrt{2^n}$ is irrational, as required.

3. (30 pts) Let $\Pi = \mathbb{R}^2 - \{0, 0\}$ be the coordinate plane without the origin. We consider Π to be a plane as in the geometry notes. Distances are the usual distances: $|PQ| = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}$ where $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$. The lines of this plane are the usual lines in the coordinate plane intersected with Π. Show that the ruler axiom cannot be true in this plane Π.

Solution:
Consider the set of points $l = \{(x, 0)|x \in \mathbb{R}\}$. The set $m = l \cap \Pi$ is a line. Suppose that the ruler axiom holds. Let $f : m \to \mathbb{R}$ be a coordinate system such that $f(1, 0) = 0$ and $f(2, 0) > 0$. Let $Q = f^{-1}(-1)$. Then the distance from $(1, 0)$ to Q is 1, as is the distance from $(1, 0)$ to $(2, 0)$. So there are 2 points on Π of distance 1 from $(1, 0)$. However, on the usual Euclidean plane $\mathbb{R}^2$, the only 2 ponts of distance 1 from $(1, 0)$ on l are $(2, 0)$ and $(0, 0)$. Since we are using the usual Euclidean distances for Π, it must be the case that $Q = (0, 0)$, a contradiction.

4. (40 total pts)

(a) (20 pts) Let T_n, $n \in \mathbb{Z}^+$, be a collection of denumerable sets. Prove that for every positive integer k the set

$$\bigcup_{n=1}^{k} T_n$$

is denumerable.

Solution:
We will induct on k. In the case where $k = 1$, this is trivial. Now assume that this is true for $k = j$, and prove it for $k = j + 1$. Observe that

$$\bigcup_{n=1}^{j+1} T_n = \left(\bigcup_{n=1}^{j} T_n \right) \cup T_{j+1}.$$

The union of j sets on the left is denumerable by the induction hypothesis, and the set T_{j+1} is also denumerable, hence so is their union.

(b) (20 pts) Is the power set $\mathcal{P}(\mathbb{Z})$ countable?

Solution:
No. First, since there is an injection from $\mathbb{Z} \to \mathcal{P}(\mathbb{Z})$ given by $n \mapsto \{n\}$, $\mathcal{P}(\mathbb{Z})$ is not finite. Second, there is no bijection from any set to its power set, so $\mathcal{P}(\mathbb{Z})$ is not denumerable (since $\mathbb{Z}$ is denumerable).

5. (40 pts) There are 21 people out of which we want to form 3 teams of 7; each team has 5 starting players and 2 alternates. In how many different ways can we do this?

Solution:
First, choose one team from the 21 people and the second team from the remaining 14 people. The remaining 7 are the remaining team. Then choose the starting players in each team. Now we have overcounted, because the order in which we chose the 3 teams does not matter, so we must divide by 3!.

$$\frac{\binom{21}{7}\binom{14}{7}\left(\binom{7}{5}\right)^3}{3!}$$

6. (60 total pts) We consider the real line $\mathbb{R}$ to be a plane as in the geometry notes. Let a be any fixed irrational number. We define the lines in this plane to be the following three subsets

$$l = \{0, a\}, \quad m = \mathbb{Q} - \{0\}, \quad n = \mathbb{R} - (\mathbb{Q} \cup \{a\}).$$

Distances are defined as usual $|PQ| = |P - Q|$.

(a) (15pts) Is the incidence axiom true or false in this situation?

False. No line contains both 0 and 1.

(b) (15 pts) Is the parallel axiom true or false in this situation?

True. These three disjoint lines exhaust the plane. Thus, since every point lies on exactly one line, any point not on one line lies on exactly one other line.

(c) (15 pts) Is the ruler axiom true or false in this situation?

False. Since l has cardinality 2, there is no bijection between it and $\mathbb{R}$.

(d) (15 pts) If we add the subset $o = \{-1, 1\}$ and call it a line, then we have a total of four lines on our plane. Is the parallel axiom true?

No. Observe that $-1 \notin n$, but both m and $\{-1, 1\}$ contain -1, and both are paralell to n.