HW7;

5.

Assume $|A| = n \in \mathbb{Z}^+ \Rightarrow$ we can enumerate A, by $\{a_1, a_2, \dots, a_n\}$.

Follow the algorithm; let $m_1 = a_1$. and $m_2 = \begin{cases} m_1 & \text{if } a_2 \geq m_1 \\ a_2 & \text{if } a_2 < m_1 \end{cases}$

and inductively, $m_{K+1} = \begin{cases} m_K & a_{K+1} \geq m_K \\ a_{K+1} & a_{K+1} < m_K \end{cases}$ $\quad K+1 \leq n$

Obviously, $m_n \in A$. Moreover, by construction, $m_n \leq a_K$ $(\forall K \leq n)$.

6. Remember the algorithm for finding the min. (and max.).

If $A \subseteq B \longrightarrow$ we can enumerate B, such that

$\underbrace{b_1, \dots, b_n}_{\in A}, \underbrace{b_{n+1}, \dots, b_p}_{\notin A} \in B = \{b_1, \dots, b_p\}$.

By the algorithm, as $m_{K+1} \leq m_K \longrightarrow m_p \leq m_n \longrightarrow \min B \leq \min A$.

Similarly for max.

10. Assume we have a surjection $f: X \longrightarrow Y$. $\quad |X| < |Y|$.

$\longrightarrow \forall y \in Y$, $\quad \overleftarrow{f}(\{y\}) \neq \phi$. Pick $x_y \in \overleftarrow{f}(\{y\}) \longrightarrow$

By putting other elements of $\overleftarrow{f}(\{y\})$ aside, (if there is any) $\longrightarrow$.

we can define a function $g: Y \longrightarrow X$, taking $y \mapsto x_y$.

Evidently, $f(x_y) = f \circ g(y) = y$, and g is injective, $Y \longrightarrow \vec{g}(Y) \subseteq X$.

But, by the pigeon-hole principle, there is no injection: $Y \rightarrow \vec{g}(Y)$.

We note that $|g(Y)| \leq |X|$

11. Suppose $f: X \longrightarrow Y$ is injective and not surjective $\longrightarrow \exists y \in Y: \forall x \in X, f(x) \neq y$

$\Rightarrow \vec{f}(X) \subseteq Y$ and $\vec{f}(X) \neq Y. \Rightarrow |\vec{f}(X)| < |Y|$.

Define $\begin{cases} g: X \rightarrow \vec{f}(X) \\ g(x) = f(x) \end{cases}$ Obviously, g is injective, but as $|\vec{f}(X)| < |X| = |Y|$

it contradicts pigeon-hole principle.

14. Define $f(a)$ = largest odd number in $\mathbb{N}_{2n}$.

The very key point is to note that $f(a) = a$, if a is odd,

and $f(a) = \frac{a}{2^\alpha}$, for some $\alpha \geq 1$, if a is even. $\rightarrow$ $\left\{\begin{array}{l}\text{So in general} \\ f(a) = \frac{a}{2^\alpha} \quad \alpha \geq 0\end{array}\right.$

Now, note that $f: \mathbb{N}_{2n} \longrightarrow$ Odd numbers in $\mathbb{N}_{2n}$, in

has its co-domain has n elements.

So, $f\big|_A : A \longrightarrow$ Odd ...

can't be injective .. $\longrightarrow \exists\, a \neq b : f(a) = f(b)$.

If $f(a) = f(b)$, $a, b \in A \longrightarrow \left\{\begin{array}{l} f(a) = \frac{a}{2^\alpha} \\ f(b) = \frac{b}{2^\beta}\end{array}\right. \longrightarrow$

$a = b\, 2^{\alpha - \beta}$

definitely, $\alpha - \beta \neq 0$ - , as $a \neq b$

$\alpha - \beta > 0 \longrightarrow b / a$

$\alpha - \beta < 0 \longrightarrow a / b$ ∎