HW5:

16-(i) surjective, and injective.

$$f_1^{-1}(x) = x + 1$$

(ii) injective, surjective.

$$f_2^{-1}(x) = x^{\frac{1}{3}}$$

(iii)By drawing the graph, we find out that it is not injective. For example, $f_3(0) = f_3(1) = 0$. So the inverse does not exist. But it is surjective.

(iv) We easily find out that $f_4(x) = (x-1)^3$. As the function $g(x) = x^3$ is surjective and injective, so is f_4. And we have

$$f_4^{-1}(x) = x^{\frac{1}{3}} + 1$$

(v)It does not assume negative numbers, so it is not surjective. But it is increasing, so it is injective.

$$f_5^{-1}(x) = Ln(x)$$

(vi) Injective, and surjective(check it for both negatives, positives, and zero).

$$f_6^{-1} = \begin{cases} x^{\frac{1}{2}} & if \quad x > 0 \\ -(-x)^{\frac{1}{2}} & if \quad x \leq 0 \end{cases}$$

18-We want to show that for each $z \in Z$, we can find an $x \in X$, such that $gof(x) = z$.

As g is surjective, there is a y such that $g(y) = z$. Moreover, as f is also surjective, we have an x such that $f(x) = g(y)$. Now, it`s done, as $gof(x) = z$.

19- Note that f(x) is a surjection. That means, for each $y \in Y$, we ca find a $x \in X$, such that $f(x) = y$.

Also note that we do not claim that x is unique. So, for each y, we can pick up one of such x`s, call it $x(y)$. As f(x) is a function, no two distinct y`s, say y_1, y_2, cannot have a common inverse image. So we can have a fuction g(x), taking each x to $x(y)$.

$$g: Y \to X$$

$$y \to x(y)$$

That is the function we sought for. As for each $y \in Y$ we have $fog(y) = f(g(y))$. But by the definition of g(y), it is in the preimage of y. So $f\big(g(y)\big) = y$.

20-

(i) $y \in f(A_1)$. This means there is an $x \in A_1$ such that $y = f(x)$. Noting that $A_1 \subseteq A_2$we find out that $x \in A_2$. So $y = f(x) \in A_2$, and this completes the proof, because every element of $f(A_1)$ is also in $f(A_2)$.
But take the function $f(x) = |x|, and\ A_1 = [1,2], A_2 = [-2,-1]$. Obviously, A_1 and A_2 have the same images, yet their intersection is empty!
If we add the condition that f(x) be one-one, we can be sure that each set has a unique pre-image, because each point has a unique pre-image.
Note that if the converse is true, f(x) must be injective. Because we can take A_1 and A_2 to be $\{x_1\}$ and $\{x_2\}$. Note that $f(\{x_1\}) = f(\{x_2\}) \Longrightarrow \{x_2\} = \{x_1\}$ means f(x) is injective.

(ii) Again, take $y \in f(A_1 \cap A_2)$ that means $\exists x \in A_1 \cap A_2, s.t, f(x) = y$. As x is in both A_1 and A_2, $f(x) \in f(A_1 \cap A_2)$.
Again look at the aforementioned example, in (i).

(iii) Assume $y \in f(A_1 \cup A_2)$. Then $\exists x \in A_1 \cup A_2, s.t, f(x) = y$. So x is in either A_1 or A_2. So its image is contained in at least of of $f(A_1)$ or $f(A_2)$. That means $y \in f(A_1) \cup f(A_2)$.
Conversely, if $y \in f(A_1) \cup f(A_2)$, $\exists x \in A_1, s.t, f(x) = y$ OR $\exists x \in A_2, s.t, f(x) = y$. That means $\exists x \in A_1 \cup A_2, s.t, f(x) = y$. That`s what we wanted: $y \in f(A_1) \cup f(A_2)$.