HW-1

Problem 1: Prove that if n is an integer, $2n + 1$ is odd.
Answer:
Let n be an integer. Suppose for the sake of contradiction that $2n + 1$ is even, i.e., that there is an integer q such that $2n+1 = 2q$. Rearranging the terms gives us that $2n-2q = 1$, or $2(n-q) = 1$. This implies that 1 is even, a contradiction. Therefore, $2n + 1$ is odd.
p. 53-54

4. (i) Show that $a|b$ and $a|c$ □ $a|(b + c)$.
Proof: Suppose that $a|b$ and $a|c$. Then there are integers p and q such that $ap = b$ and $aq = c$. Thus $ap + aq = b + c$, but $ap + aq = a(p + q)$, so $a|(b + c)$, as required.
(ii) Show that $a|b$ or $a|c$ □ $a|bc$. Proof:
Case 1: $a|b$. Then there is an integer q such that $aq = b$. Then $bc = aqc$, so $a|bc$.
Case 2: $a|c$. Then there is an integer q such that $aq = c$. Then $bc = baq$, so $a|bc$.

5. The following are necessary for an integer n to be divisible by 6. (i),(v),(vi),(vii)
The following are sufficient for an integer n to be divisible by 6. (iii), (iv), (v), (vi)
6. (i) $(1+0)a = 1.a\text{-}0.a = a\text{-}0.a=a$
By cancellation, we find out that $0.a=0$.
Similarly, $a.0 =0$.
(ii) Show that $(-a)b = -ab = a(-b)$.
Proof: $-a$ is the unique number such that $a+(-a) = 0$. Multiplying both sides of this equality by b gives us that $(a + (-a))b = 0 \cdot b = 0$ (by (i)). Using the distributive property, we have $ab + (-a)b = 0$. On the other hand, $-ab$ is the unique number such that $ab + (-ab) = 0$. Hence, by uniqueness, $(-a)b = -ab$. To show the second equality, begin with $b + (-b) = 0$, and observe that $a(b + (-b)) = a \cdot 0$ □ $ab + a(-b) = 0$ □ $ab = a(-b)$.
(iii) Show that $(-a)(-b) = ab$.

Observe that $0 = (a+(-a))(b+(-b)) = ab+(-a)b+a(-b)+(-a)(-b)$. Using (ii), we see that this last is $ab+(-ab)+(-ab)+(-a)(-b) = 0+(-ab)+(-a)(-b)$. But ab is the unique number such that $(-ab) + ab = 0$. Thus, by uniqueness, $(-a)(-b) = ab$.

7. Show that n2 is even implies that n is even. We may assume that an integer is odd if and only if it can be written as $2q + 1$ for some integer q.
Proof: We will prove the contrapositive, i.e. that if n is odd, so is n2. For, suppose that there is some integer q such that $n = 2q+1$. Then n2 = $(2q+1)$2 = $4q$2 + $4q$ + 1 = 2(2q2 + 2q) + 1. Taking p = 2q2 + 2q, n2 = $2p + 1$, and hence is odd.

8.(i) Suppose that x is a real number. Show that $x2 - x - 2 = 0$ □ $x = -1$ or $x = 2$.

Proof: Observe that $x2 - x - 2 = (x + 1)(x - 2)$.

" □: Suppose that $x = -1$. Then $x + 1 = 0$, and hence $(x + 1)(x - 2) = 0$. Similarly, if $x = 2$, then $x - 2 = 0$, so $(x + 1)(x - 2) = 0$.

" □: Suppose that $(x + 1)(x - 2) = 0$. By proposition 4.4.1, $x + 1 = 0$ or $x - 2 = 0$. Hence, $x = -1$ or $x = 2$.

9. Prove that there is no largest integer.

Proof: Suppose for a contradiction that there is a largest integer n. Observe that $0 < 1$. Adding n to both sides shows that $n < n+ 1$, a contradiction.

10. This proof shows that if there is a largest integer, then that integer is 1. Since there is no largest integer (number 9), this statement is certainly true, since the hypothesis is false.

11. Prove that there is no smallest positive real number.

Proof: Suppose for the sake of contradiction that x is the smallest positive real number. Observe that $0 < x$. Multiplying both sides of this inequality by 1/2 yields that $0 < x/2$. Adding $x/2$ to both sides of this inequality yields that $x/2 < x$. To sum up, $0 < x/2 < x$, in other words, $x/2$ is a smaller positive real number than x, which is a contradiction.