

MATH 200, Lec 2

Final Exam

December 20, 2006

Name: _____ ID: _____

Question:	1	2	3	4	5	6	7	8	Extra	Total
Points:	10	12	10	10	10	10	10	10	0	82
Score:										

There are 9 problems in this exam. Make sure that you have them all.

Do all of your work in this exam booklet, and cross out any work that the grader should ignore. You may use the backs of pages, but indicate what is where if you expect someone to look at it. **Books and discussions with friends are not permitted.** You may use **one** sheet of handwritten notes, provided you turn it in with the exam. A set of enchanted notes (like Tom Riddle's diary in *Harry Potter and the Chamber of Secrets*) is permitted, but frowned upon because their use tends to have unforeseen consequences.

You have two hours or so to complete this exam.

1. (a) 5 points You probably have seen the following statement on a truck

If you can't see my mirrors, then I can't see you.

Write a logically equivalent statement that does not use any negatives.

- (b) 5 points A subset U of \mathbb{R} is called an *open set* when the following property holds:

For each point $x \in U$, there is a $\delta > 0$ so that for every z , if $|x - z| < \delta$ then $z \in U$.

Without using any negatives except \neq , write a definition of what it means for U not to be an open set. You may write this symbolically or in words, as you prefer, but write it carefully and correctly.

2. You have 5 rabbits named Flippy, Floppy, Flappy, Floopy, and George; two turtles named Terrence and Tabitha; and three foxes named Xavier, Xam, and Xue. You can tell all of your animals apart. You want to take three of your animals to visit your sick aunt Bertha (she just loves animals), but you only have one cage– the cage holds exactly three animals.

(a) 4 points How many different trios of animals could you pick? (The animals needn't be the same species, but they might be.) Justify your answer.

(b) 4 points If you put a fox in a cage with a rabbit, the fox will eat the rabbit. The rabbits and the turtles just ignore one another, and the turtles and the foxes get along famously. How many bad choices of animals could you make (that is, how many trios contain at least one fox and at least one rabbit)? Again, justify your answer.

(c) 4 points Finally, how many different sets of three animals can you put in your cage safely, assuming you can't put foxes and rabbits together?

3. 10 points Prove that there is no rational number whose square is 8.
You may assume that for any integer a , if a^2 is even, then a is also even.
Be careful; there is a minor difference with the proof for $\sqrt{2}$.

4. 10 points Prove that for all integers $n > 1$,

$$1 + 4 + 7 + \dots + (3n - 2) = \frac{n(3n - 1)}{2}$$

(You might find induction helpful.)

5. 10 points Let $\triangle ABC$ be a triangle. Prove that $m\angle A = m\angle B = m\angle C$ if and only if $|AB| = |BC| = |CD|$.

6. 10 points What is the last digit of 3^{100} ?

Don't try to multiply this out: the answer has 47 digits.

7. Consider the following axioms describing a nonempty set S of people.

DEFINITION. A club is a nonempty set of people. If a person p is a member of club C , we write $p \in C$. Clubs are determined by their members; that is, two clubs with exactly the same members are the same club.

AXIOM 1. Every person in S is the member of at least one club.

AXIOM 2. For every club C , there is exactly one club \bar{C} which shares no members with it. This club is called the nemesis of C .

AXIOM 3. For each pair of people, there is exactly one club to which they both belong.

(a) 5 points Show that each person is a member of at least two clubs.

(b) 5 points Show that there are at least four people in S .

8. (a) 5 points Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$.
Is f a bijection? Justify your answer.

- (b) 5 points Let $g : \mathbb{Z} \rightarrow \mathbb{Z}$ be given by $g(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$.
Is g a bijection? Justify your answer.

Extra Credit (up to 10 points)

There is a statement in each of the four boxes, but three of the statements are false, and one is true. Exactly one of the boxes is worth 4 points; the others are worth no points. If you place an X in the box worth four points, you may have them. If you put it in a worthless box, you get nothing. If you put an X in more than one box, I will **deduct** 10 points, so don't do that.

Box A	Box B	Box C	Box D
This box is worth no points	Box C is worth 4 points	Box D is worth 4 points	The statement in Box C is false

For 6 additional points, you must give a proof that the box you picked was the one worth points, and that there is only one such box.