

Axioms and Definitions for Geometry

UNDEFINED TERMS . The plane is our universe of discourse, and points and lines are subsets of the plane. A line is a set of points with properties as defined by the axioms. The distance between any two points A and B is a number denoted by $|AB|$, again with properties as specified by the axioms.

INCIDENCE AXIOM.

1. For any two distinct points, there is a unique line that contains these two points.
2. Every line contains at least two distinct points.
3. For any line, there exists a point not on this line.

DEFINITION. Two lines l and m are said to be transverse if they are distinct ($l \neq m$) and have at least one point in common. Two lines are parallel if they are not transverse.

THE PARALLEL AXIOM. For any line l and a point P not on l , there exists a unique line containing P and parallel to l .

THE RULER AXIOM. Let l be any line. Then there is a bijection $f : l \rightarrow R$ such that, for any two points A, B on l , the distance between A and B, $|AB|$, is given by $|f(A) - f(B)|$. This bijection f is called a coordinate system on l .

DEFINITION. If A, B, and C are points on a line l , we say B is between A and C if there is a coordinate system f on l for which $f(A) < f(B) < f(C)$. The set of all points on l that are between between A and C is called the line segment \overline{AB} .

DEFINITION. Let A, B and C be three distinct points on a line l . We say that A and C are on opposite sides of B if B is between A and C . If A and B are not on opposite sides of C , we say A and B are on the same side of C .

DEFINITION. If l is a line and V and A are distinct points on l , we define the ray \overrightarrow{VA} to be all of the points on l that are on the same side of V as A.

BINOMIAL THEOREM:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k.$$

You may write answers in terms of factorials and exponents.

1. Suppose that a computer password is 6 letters long and it contains only lower-case letters (from the 26-letter alphabet).
 - (a) (5 pts) Find a formula for how many different such passwords there are. Note that the password can be any combination of letters (such as abacad).
 - (b) (5 pts) How many possibilities are there if no letter is repeated?
2. A deck of cards contains 52 distinct cards.
 - (a) (5 pts) How many different 7-card hands are there (i.e. how many different ways are there to draw 7 cards from the deck if the order doesn't matter)?
 - (b) (5 pts) Suppose that there are a total of 4 aces in the deck. How many different 7-card hands are there with exactly 2 aces?
3. (5 pts) Write the number $0.123123123\dots$ as a fraction.
4. (10 pts) For each $n \in \mathbb{Z}^+$, let S_n be a denumerable set. Suppose that for each $n \neq m$, $S_n \cap S_m = \emptyset$. Show that

$$\bigcup_{n \in \mathbb{Z}^+} S_n$$

is also denumerable.

5. (5 pts) Suppose that $f : I \rightarrow \mathbb{R}$ is a coordinate system and $c \in \mathbb{R}$. Show that $g(A) = c - f(A)$ is a coordinate system.
6. For each of the interpretations of the terms point, line, and distance given below, determine if they are consistent with the indicated axioms.
 - (a) Points are elements of the set $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. The distance between two points (x_1, y_1) and (x_2, y_2) is given by

$$\sqrt{(x_1 - x_2)^2 + \left(\ln\left(\frac{y_1}{y_2}\right)\right)^2}.$$

Lines are just intersections between the usual lines in \mathbb{R}^2 and the plane defined above.

- i. (3 pts) Prove that the incidence axiom is satisfied or show that it isn't.
 - ii. (3 pts) Prove that the parallel axiom is satisfied or show that it isn't.
- (b) Points are real numbers, and lines are integers. The distance between two real numbers is the absolute value of their difference.
 - i. (3 pts) Prove that the incidence axiom is satisfied or show that it isn't.
 - ii. (3 pts) Prove that the parallel axiom is satisfied or show that it isn't.
 - iii. (3 pts) Prove that the ruler axiom is satisfied or show that it isn't.