Part 1

Non-compact hyperkähler manifolds

CHAPTER 1

Holonomy

The material of this section is mainly standard Riemannian geometry. Further details may be found in [**J**], [**P**], or [**S**].

1.1. Berger's classification

Let (M, g) be a connected Riemannian manifold of dimension n, and let $\gamma : [0,1] \to M$ be a piecewise smooth path joining a point p to a point q. Parallel transport with respect to the Levi-Civita connection gives a linear map

$$P_\gamma : T_pM \to T_qM.$$

Suppose that $p = q$, so that γ is a closed loop. Since the metric is preserved by the connection, $\nabla g = 0$, the map P_γ belongs to the orthogonal group $\mathrm{O}(T_pM)$.

DEFINITION 1.1. The holonomy group of (M, g) at p is

$$\mathrm{Hol}_p(g) := \{P_\gamma | \text{all loops } \gamma \text{ based at } p\}.$$

The restricted holonomy group is

$$\mathrm{Hol}_p^0(g) := \{P_\gamma | \text{all null-homotopic loops } \gamma \text{ based at } p\}.$$

REMARK 1.2. After choosing an orthonormal basis for T_pM, we can identify $\mathrm{O}(T_pM)$ with $\mathrm{O}(n)$, and thus regard $\mathrm{Hol}_p^0(g) \subset \mathrm{Hol}_p(g)$ as subgroups of $\mathrm{O}(n)$ (well-defined, up to conjugation). Moreover, changing the basepoint p also changes these subgroups only by conjugation, and so we can talk about $\mathrm{Hol}^0(g)$ and $\mathrm{Hol}(g)$.

The restricted holonomy group $\mathrm{Hol}^0(g)$ is a closed connected Lie subgroup of $\mathrm{O}(n)$, and hence compact. It sits inside $\mathrm{Hol}(g)$ as a normal subgroup, and is the connected component of the identity. In most of the cases we will encounter $\mathrm{Hol}(g)$ is also a closed Lie subgroup of $\mathrm{O}(n)$, though it is actually an open problem to determine precisely when this happens, for M compact [**S**].

There is a surjection from the fundamental group

$$\pi_1(M) \twoheadrightarrow \mathrm{Hol}(g)/\mathrm{Hol}^0(g)$$

so $\mathrm{Hol}(g) = \mathrm{Hol}^0(g)$ if M is simply-connected.

We will present a list of possible holonomy groups of Riemannian manifolds, though we want to make some simplifying assumptions that will shorten the list. So first we will discuss some particular cases.

For a product of manifolds $(M_1 \times M_2, g_1 \times g_2)$, with the product metric, the holonomy group is also a product

$$\mathrm{Hol}(g_1 \times g_2) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2).$$

DEFINITION 1.3. The Riemannian manifold (M, g) is locally reducible if every point has a neighbourhood isometric to a product, and irreducible otherwise.

REMARK 1.4. If (M, g) is irreducible then $\mathrm{Hol}(g)$ acts irreducibly on $\mathbb{R}^n$.

DEFINITION 1.5. The Riemannian manifold (M, g) is a symmetric space if for every point $p \in M$ there exists an isometry $s_p : M \to M$ such that $s_p^2 = \mathrm{Id}$ (s_p is an involution) and p is an isolated fixed point of s_p.

For example, s_p could be reflection around the point p on a sphere S^n. Note that p need not be a unique fixed point of s_p.

THEOREM 1.6. *Assume that M is simply-connected. Define the subgroup*

$$G := \{s_q \circ s_r | q, r \in M\} \subset \mathrm{Isom}(M)$$

of the isometry group of M. Then G is a connected Lie group which acts transitively on M, and $M = G/H$ where H is the (closed, connected, Lie) subgroup of G fixing some point $p \in M$. Moreover, the holonomy group $\mathrm{Hol}_p(g)$ can be identified with H.

EXAMPLE 1.7. Some examples of symmetric spaces, written in the above way, are Euclidean space, the sphere, hyperbolic space, complex projective space

$$\mathbb{R}^n = \mathbb{R}^n/\{0\}, \qquad S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n), \qquad H^n = \mathrm{SO}(n,1)/\mathrm{SO}(n),$$
$$\mathbb{CP}^n = (\mathrm{U}(n+1)/\mathrm{U}(1))/\mathrm{U}(n) = \mathrm{U}(n+1)/\mathrm{U}(1) \times \mathrm{U}(n).$$

(In the last example $G = \mathrm{U}(n+1)/U(1)$ and $H = \mathrm{U}(n)$.)

Simply-connected symmetric spaces were classified by Elie Cartan in 1927 using the theory of Lie groups. Consequently we know which groups can arise as holonomy groups of symmetric spaces. In particular, every connected compact simple Lie group occurs as a holonomy group, with holonomy representation on $\mathbb{R}^n$ given by the adjoint representation.

DEFINITION 1.8. A Riemannian manifold (M, g) is locally symmetric if for every point $p \in M$ there exists a local isometry $s_p : U_p \to U_p$, where U_p is a neighbourhood of p, such that $s_p^2 = \mathrm{Id}$ and p is an isolated fixed point of s_p.

THEOREM 1.9. *A locally symmetric manifold is locally isometric to a unique simply-connected symmetric space. Moreover, (M, g) is locally symmetric if and only if the Riemannian curvature tensor is covariantly constant, $\nabla R = 0$.*

REMARK 1.10. It follows that the restricted holonomy of a locally symmetric manifold is the same as the holonomy of some symmetric space.

We can now state Berger's theorem, which was proved in 1955.

THEOREM 1.11 (Berger). *Suppose the Riemannian manifold (M, g) is simply-connected (and hence oriented), irreducible, and not locally symmetric. Then its holonomy group $\mathrm{Hol}(g)$ is one of the following:*

1. *generic:* $\mathrm{SO}(n)$,
2. *Kähler:* $\mathrm{U}(m) \subset \mathrm{SO}(2m)$, *for $n = 2m$ and $m \geq 2$,*
3. *Calabi-Yau:* $\mathrm{SU}(m) \subset \mathrm{SO}(2m)$, *for $n = 2m$ and $m \geq 2$,*
4. *hyperkähler:* $\mathrm{Sp}(m) \subset \mathrm{SO}(4m)$, *for $n = 4m$ and $m \geq 2$,*
5. *quaternion-Kähler:* $\mathrm{Sp}(m)\mathrm{Sp}(1) := \mathrm{Sp}(m) \times \mathrm{Sp}(1)/\mathbb{Z}_2 \subset \mathrm{SO}(4m)$, *for $n = 4m$ and $m \geq 2$,*
6. *G_2-manifolds:* $G_2 \subset \mathrm{SO}(7)$, *for $n = 7$,*
7. $\mathrm{Spin}(7)$*-manifolds:* $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$, *for $n = 8$.*

Note that $\mathrm{Sp}(m) \subset \mathrm{GL}(m, \mathbb{H}) \subset \mathrm{GL}(4m, \mathbb{R})$ *is the subgroup of* $m \times m$*-matrices with quaternion entries which preserve the inner product*

$$(v, w) := \mathrm{Re} \sum_j \bar{v}_j w_j,$$

and is sometimes called the quaternionic unitary group. When $m = 1$*, we recover* $\mathrm{SU}(2) = \mathrm{Sp}(1)$. *Four-dimensional manifolds with this holonomy are still called hyperkähler manifolds, though they are also Calabi-Yau manifolds.*

REMARK 1.12. The groups $\mathrm{O}(n)$, $\mathrm{U}(m)$, and $\mathrm{Sp}(m)\mathrm{Sp}(1)$ can be regarded as the metric preserving automorphism groups of $\mathbb{R}^n$, $\mathbb{C}^m$, and $\mathbb{H}^m$ respectively. The groups $\mathrm{SO}(n)$, $\mathrm{SU}(m)$, and $\mathrm{Sp}(m)$ can be regarded as the determinant one subgroups of the former groups. In some sense, one can also add $\mathrm{Spin}(7)$ (which acts on the octonians $\mathbb{O}$) to the first three groups, and G_2 (which acts on the imaginary octonians $\mathrm{Im}\mathbb{O}$) to the second three groups, as these actions also preserve certain structures.

There are many examples of Kähler manifolds, and obviously of manifolds with generic holonomy. It is also not difficult to find symmetric spaces with holonomy $\mathrm{Sp}(m)\mathrm{Sp}(1)$. However, one can show that metrics with holonomy $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, G_2, and $\mathrm{Spin}(7)$ must be Ricci-flat (we will illustrate a proof this for the first two groups shortly). In particular, symmetric spaces cannot have these holonomy groups, because although symmetric spaces are Einstein (their Ricci curvature is a scalar multiple of the metric), if they are Ricci-flat they must be flat. Perhaps for this reason it took many years to find examples of Riemannian manifolds with these holonomy groups.

We will now give an outline (following [**J**]) of how Berger's theorem is proved.

PROOF. Let $\mathfrak{h}_p$ be the Lie algebra of $\mathrm{Hol}^0_p(g)$. Then $\mathfrak{h}_p$ is a Lie subalgebra of $\mathfrak{o}(T_pM)$, which is isomorphic to $\Lambda^2 T^*_p M$ (by using the metric). Suppose that X and Y are commuting vector fields, $[X, Y] = 0$, in a neighbourhood of p. The we can form a small parallelogram with one corner at p, and with edges of length ϵ in the directions of X and Y. More precisely, the edges will be given by following integral curves of X and Y a distance ϵ; since X and Y commute these integral curves will form a closed loop γ.

Now parallel transport P_ϵ around this loop will lie in $\mathrm{Hol}^0_p(g)$. The instrinsic geometric definition of the curvature at p is as the derivative of P_ϵ

$$R_p(X, Y)Z = \lim_{\epsilon \to 0} \frac{P_\epsilon Z - Z}{\epsilon}.$$

Therefore $R_p(X, Y)$ lies in the $\mathfrak{h}_p \subset \Lambda^2 T^*_p M$. Furthermore, the symmetries of the curvature tensor imply that R_p lies in $\mathrm{Sym}^2 \mathfrak{h}_p$.

There is a converse to this result, known as the Ambrose-Singer theorem. Any null-homotopic loop can be decomposed into lassoos, ie. a loop consisting of a path from p to the corner q of a small parallelogram, followed by the circumference of the parallelogram, followed by the first path in the opposite direction. If we denote by P parallel transport along the path from p to q, and by X and Y the (local) commuting vector fields defining the parallelogram, then it follows from this decomposition that $\mathfrak{h}_p$ will be generated by elements of the form $P^{-1} \circ R_q(X, Y) \circ P$. Thus the curvature $R(X, Y)$ sits inside some smaller subspace of $\Lambda^2 T^*_p M$ if and only if we have a reduction of holonomy.

Suppose that $H \subset \mathrm{SO}(n)$ is a closed connected Lie subgroup acting irreducibly on $\mathbb{R}^n$, with Lie algebra $\mathfrak{h}$. Already from Lie theory we get a list, albeit a long list, of possible groups. Berger used two criteria to rule out most of the groups on this list. Let $\mathcal{R}^H$ denote the elements of $\mathrm{Sym}^2\mathfrak{h}$ which satisfy the first Bianchi identity. The curvature R of a manifold with holonomy group H must lie in $\mathcal{R}^H$ at each point. Therefore

(1) $\mathcal{R}^H$ must be large enough to generate $\mathfrak{h}$, as the Ambrose-Singer theorem requires,
(2) the covariant derivative ∇R would lie in $(\mathbb{R}^n)^* \otimes \mathcal{R}^H$, and it must also satisfy the second Bianchi identity; the space of such tensors must be non-trivial, as otherwise $\nabla R = 0$ and we'd have a locally symmetric space.

The only groups to satisfy both criteria are those on Berger's list. What is perhaps surprising is that all of the groups on Berger's list were eventually realized as holonomy groups of some Riemannian manifolds. □

REMARK 1.13. It was recognized that Berger's list consists precisely of those groups acting transitively on the sphere $S^{n-1} \subset \mathbb{R}^n$. In 1962 Simons gave a direct proof of this fact, and hence a shorter (though still difficult) proof of Berger's theorem.

1.2. Constant tensors

The frame bundle F is the principal $\mathrm{GL}(n,\mathbb{R})$ bundle on M, whose fibre over $p \in M$ consists of bases for T_pM. A connection on the tangent bundle TM is equivalent to a connection on F, and we denote both by ∇. On a Riemannian manifold (M, g) there is a reduction of the frame bundle to a $\mathrm{O}(n)$ bundle, consisting of orthonormal frames. Associated to any representation V of $\mathrm{GL}(n,\mathbb{R})$ (or of $\mathrm{O}(n)$ in the Riemannian case) there is a bundle

$$E := F \times_{\mathrm{GL}(n,\mathbb{R})} V$$

with connection ∇^E induced from ∇. Some examples include

- the standard representation on $\mathbb{R}^n$ gives the tangent bundle TM,
- the dual representation $(\mathbb{R}^n)^*$ gives the cotangent bundle T^*M,
- the representation

$$V_{k,l} := \bigotimes^k \mathbb{R}^n \otimes \bigotimes^l (\mathbb{R}^n)^*$$

gives the tensor bundle $\bigotimes^k TM \otimes \bigotimes^l T^*M$.

We can also construct exterior and symmetric powers of TM and T^*M; indeed all tensor bundles arise in this way.

THEOREM 1.14. *Let (M, g) be a Riemannian manifold. The holonomy group $\mathrm{Hol}_p(g) \subset \mathrm{O}(T_pM)$ acts on the fibre $E_p \cong V$. Suppose that $S \in C^\infty(E)$ is a covariant constant section, $\nabla^E S = 0$. Then $S|_p \in E_p$ is fixed by $\mathrm{Hol}_p(g)$. Conversely, if $S_p \in E_p$ is fixed by $\mathrm{Hol}_p(g)$ then there exists a unique constant section $S \in C^\infty(E)$ with $S|_p = S_p$.*

PROOF. If S is constant then it is preserved by parallel transport around a loop. Since parallel transport around loops based at p generates the holonomy group $\mathrm{Hol}_p(g)$, S_p is fixed by this group.

Conversely, given S_p fixed by $\mathrm{Hol}_p(g)$, we can construct a constant section of E by parallel transporting S_p to other points. If this were not well-defined there would exists two paths γ_1 and γ_2 from p to some point q, and parallel transport around $\gamma_2^{-1} \circ \gamma_1$ would not fix S_p, a contradiction. □

EXAMPLE 1.15. The metric is a covariantly constant section of the bundle $\mathrm{Sym}^2 T^*M$, and therefore the holonomy group $\mathrm{Hol}_p(g)$ preserves the metric at the point p. This means $\mathrm{Hol}_p(g) \subset \mathrm{O}(T_pM)$.

EXAMPLE 1.16. Suppose that (M, g) admits an almost complex structure I, a smooth section of $\mathrm{End}TM$ such that $I^2 = -1$. If I is constant (which is not generally the case) then $\mathrm{Hol}_p(g)$ fixes I_p and therefore is a subgroup of $\mathrm{GL}(m, \mathbb{C})$. Therefore

$$\mathrm{Hol}(g) \subset \mathrm{O}(2m) \cap \mathrm{GL}(m, \mathbb{C}) = \mathrm{U}(m)$$

(M is a Kähler manifold).

EXAMPLE 1.17. On a Calabi-Yau manifold of complex dimension m, there is a covariantly constant section Ω of the canonical bundle $K = \Lambda^{m,0}T^*M$. The holonomy group must fix this tensor, and hence $\mathrm{Hol}(g) \subset \mathrm{SU}(m)$.

Hyperkähler manifolds have holonomy group $\mathrm{Sp}(m)$. We will discuss the constant tensors on a hyperkähler manifold in the next chapter, after describing some of the basics of quaternion algebra.

COROLLARY 1.18. *Let (M, g) be a Riemannian manifold and let $G \subset \mathrm{GL}(T_pM)$ be the subgroup that fixes $S|_p$ for all constant tensors S on M. Then $\mathrm{Hol}_p(g) \subset G$. Usually we have equality, for instance if $\mathrm{Hol}_p(g)$ is compact and connected.*

CHAPTER 2

Basic results

2.1. Quaternion algebra

The main source for this section is the notes from a course by Nigel Hitchin, Oxford 2001.

We denote by $\mathbb{H}$ the quaternions

$$\mathbb{H} = \{x = x_0 + x_1 i + x_2 j + x_3 k\} \cong \mathbb{R}^4$$

with multiplication

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k, \text{ etc.}$$

Conjugation is defined by

$$\bar{x} = x_0 - x_1 i - x_2 j - x_3 k.$$

Note that $\overline{pq} = \bar{q}\bar{p}$. We also define

$$|x|^2 = x\bar{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

Let $\mathbb{H}^m$ be the quaternion vector space, the set of m-tuples $(q^1, \dots, q^m)$, with $q^\alpha = x_0^\alpha + x_1^\alpha i + x_2^\alpha j + x_3^\alpha k$, and with scalar multiplication on the right

$$q(q^1, \dots, q^m) = (q^1\bar{q}, \dots, q^m\bar{q}).$$

We define $\mathrm{GL}(m, \mathbb{H})$ to be the group of $\mathbb{R}$-linear transformations commuting with scalar multiplication on the right by i, j, and k (and therefore commuting with scalar multiplication by all quaternions). One can think of $\mathrm{GL}(m, \mathbb{H})$ as invertible $m \times m$-matrices with values in the quaternions, action by left multiplication on $\mathbb{H}^m$.

We can define a metric and two-forms on $\mathbb{H}^m$ by

$$g := \sum_{\alpha=1}^{m} (dx_0^\alpha)^2 + (dx_1^\alpha)^2 + (dx_2^\alpha)^2 + (dx_3^\alpha)^2$$

$$\omega_1 := \sum_{\alpha=1}^{m} dx_0^\alpha \wedge dx_1^\alpha + dx_2^\alpha \wedge dx_3^\alpha$$

$$\omega_2 := \sum_{\alpha=1}^{m} dx_0^\alpha \wedge dx_2^\alpha - dx_1^\alpha \wedge dx_3^\alpha$$

$$\omega_3 := \sum_{\alpha=1}^{m} dx_0^\alpha \wedge dx_3^\alpha + dx_1^\alpha \wedge dx_2^\alpha.$$

Note that

$$g + \omega_1 i + \omega_2 j + \omega_3 k = \sum_{\alpha=1}^{m} d\bar{q}^\alpha \otimes dq^\alpha$$

where we use quaternionic multiplication when expanding the right-hand-side.

If we identify $\mathbb{H}^m$ with $\mathbb{R}^{4m}$ then g becomes the usual Euclidean metric. Right multiplication by i, j, and k give complex structures I, J, and K on $\mathbb{R}^{4m}$. More generally, for any triple (a, b, c) of real numbers satisfying $a^2 + b^2 + c^2 = 1$ we get a complex structure $aI + bJ + cK$ on $\mathbb{R}^{4m}$, and g is Hermitian with respect to all of these complex structures. Thus there is a two-sphere S^2 of complex structures compatible with the metric, a fact that we will exploit later in the construction of twistor spaces.

The quaternionic unitary group $\mathrm{Sp}(m)$ can be described in any of the following ways:

- the subgroup of $\mathrm{GL}(m, \mathbb{H})$ preserving g,
- the subgroup of $\mathrm{GL}(4m, \mathbb{R})$ preserving g and the complex structures I, J, and K,
- the subgroup of $\mathrm{GL}(4m, \mathbb{R})$ preserving g and the two-forms ω_1, ω_2, and ω_3.

Of course any two of g, I, and ω_1 determine the third (and likewise for the other complex structures and two-forms). If we denote quaternionic $m \times m$-matrices by $M_m(\mathbb{H})$ then

$$\mathrm{Sp}(m) := \{A \in M_m(\mathbb{H}) | A\bar{A}^t = \mathrm{Id}\}.$$

The group $\mathrm{Sp}(m)$ is a compact connected simply-connected semi-simple Lie group of dimension $2m^2 + m$.

REMARK 2.1. The first description says that g is the subgroup of $\mathrm{GL}(m, \mathbb{H})$ preserving g. In fact g is also preserved by a larger group, namely

$$\mathrm{Sp}(m).\mathrm{Sp}(1) := \mathrm{Sp}(m) \times \mathrm{Sp}(1)/\mathbb{Z}_2$$

where $\mathbb{Z}_2$ is the subgroup generated by $(-1, -1)$, though this is not a subgroup of $\mathrm{GL}(m, \mathbb{H})$. This holonomy group leads to quaternion-Kähler manifolds.

We can also identify $\mathbb{H}^m = \mathbb{R}^{4m}$ with $\mathbb{C}^{2m}$ by taking $(z_1, \ldots, z_{2m})$ as coordinates on $\mathbb{C}^{2m}$ where $z_{2\alpha-1} = x_0^\alpha + ix_1^\alpha$ and $z_{2\alpha} = x_2^\alpha + ix_3^\alpha$. This corresponds to choosing I as the complex structure on $\mathbb{R}^{4m}$, by no means a canonical choice. On $\mathbb{C}^{2m}$ we have

$$g = \sum_{\alpha=1}^{2m} |dz_\alpha|^2, \ \omega_1 = \frac{i}{2} \sum_{\alpha=1}^{2m} dz_\alpha \wedge d\bar{z}_\alpha, \ \text{and } \sigma := \omega_2 + i\omega_3 = \sum_{\alpha=1}^{m} dz_{2\alpha-1} \wedge dz_{2\alpha}$$

which are the standard Hermitian metric, Hermitian form, and complex symplectic form. Note also that

$$\frac{1}{m!}\sigma^m = dz_1 \wedge \cdots \wedge dz_{2m}$$

is the standard holomorphic volume form on $\mathbb{C}^{2m}$. Since $\mathrm{Sp}(m)$ preserves this volume form, we have $\mathrm{Sp}(m) \subset \mathrm{SU}(2m)$. There is equality only when $m = 1$ (for $m > 1$, $\dim\mathrm{Sp}(m) = 2m^2 + m$ is strictly smaller than $\dim\mathrm{SU}(2m) = 4m^2 - 1$).

That completes our discussion of the $\mathrm{Sp}(m)$-invariant tensors on $\mathbb{R}^{4m}$. If a Riemannian manifold (M, g) has holonomy group $\mathrm{Hol}(g)$ contained in $\mathrm{Sp}(m)$ then each of these tensors must correspond to a covariantly constant tensor on M. Therefore we have constant metric g, constant almost complex structures I, J, and K, and constant two-forms ω_1, ω_2, and ω_3. We will see shortly that the almost complex structures must be integrable.

DEFINITION 2.2. We call the Riemannian manifold (M, g) with the additional structures $\{I, J, K, \omega_1, \omega_2, \omega_3\}$ a hyperkähler manifold. It has holonomy group contained in $\mathrm{Sp}(m)$. If $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ then we say (M, g) is irreducible.

REMARK 2.3. The name hyperkähler refers to the fact that (M, g) is a Kähler manifold in many different ways, as we will see shortly.

2.2. Characterizations of hyperkähler manifolds

Let us first consider a single almost complex structure on (M, g). So let (M, g, I, ω) be an almost Hermitian manifold. In other words, I is an almost complex structure, g is Hermitian with respect to I ($g(IX, IY) = g(X, Y)$), and ω is the two-form defined by $\omega(X, Y) := g(X, IY)$.

Let N be the Nihenhuis tensor

$$N(X, Y) = 2([IX, IY] - [X, Y] - I[X, IY] - I[IX, Y])$$

which gives the obstruction to the integrability of I. If N vanishes identically we say that I is torsion free, and hence I is a complex structure by the Newlander-Nirenberg theorem. The exterior derivative of the two-form ω is given by

$$\begin{aligned} 3d\omega(X, Y, Z) &= X(\omega(Y, Z)) + Y(\omega(Z, X)) + Z(\omega(X, Y)) \\ &\quad -\omega([X, Y], Z) - \omega([Z, X], Y) - \omega([Y, Z], X). \end{aligned}$$

LEMMA 2.4. *On an almost Hermitian manifold (M, g, I, ω) we have*

$$4g((\nabla_X I)Y, Z) = 6d\omega(X, IY, IZ) - 6d\omega(X, Y, Z) + g(N(Y, Z), IX).$$

PROOF. We can write

$$\begin{aligned} g((\nabla_X I)Y, Z) &= g(\nabla_X(IY), Z) - g(I(\nabla_X Y), Z) \\ &= g(\nabla_X(IY), Z) + g(\nabla_X Y, IZ) \end{aligned}$$

and

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) \\ &\quad +g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X). \end{aligned}$$

The rest of the proof is simply algebra, using the formulae for N and $d\omega$ above. □

COROLLARY 2.5. *An almost Hermitian manifold (M, g, I, ω) is Kähler (that is, I is integrable and ω is d-closed) if and only if $\nabla I = 0$.*

PROOF. If I is integrable ($N = 0$) and $d\omega = 0$ then it is clear from the lemma that we also have $\nabla I = 0$.

Conversely, we always have $\nabla g = 0$. If $\nabla I = 0$ then $\nabla\omega = 0$, since ω is defined solely in terms of g and I. Since ω is constant, it is also d-closed ($d\omega \in C^\infty(\Lambda^3 T^*)$ is the skew-symmetrization of $\nabla\omega \in C^\infty(T^* \otimes \Lambda^2 T^*)$). The lemma now implies $N = 0$, so I is integrable, and (M, g, I, ω) is a Kähler manifold. □

REMARK 2.6. The definition of a Kähler manifold requires $d\omega = 0$, but this implies the stronger condition $\nabla\omega = 0$.

REMARK 2.7. Let $2m$ be the real dimension of M. Following Salamon [**S**], denote by $\lambda^{p,q}$ and Λ^k the representations $\Lambda^p(\mathbb{C}^m)^* \otimes \Lambda^q(\overline{\mathbb{C}^m})^*$ and $\Lambda^k(\mathbb{R}^{2m})^*$ of $\mathrm{U}(m)$ (the latter is induced by the inclusion $\mathrm{U}(m) \subset \mathrm{SO}(2m)$). Let $\lambda_0^{p,q}$ denote the space of primitive forms under the Lefschetz action. Denote by $[\lambda^{p,q}]$ (for $p \neq q$) and $[\lambda^{p,p}]$ the real vector spaces such that

$$[\lambda^{p,q}] \otimes_{\mathbb{R}} \mathbb{C} = \lambda^{p,q} \oplus \lambda^{q,p} \qquad \text{and} \qquad [\lambda^{p,p}] \otimes_{\mathbb{R}} \mathbb{C} = \lambda^{p,p}.$$

Note that we have the following decompositions into irreducible representations

$$\begin{aligned} \Lambda^1 &= [\lambda^{0,1}] \\ \Lambda^2 &= [\lambda^{0,2}] \oplus [\lambda_0^{1,1}] \oplus [\lambda^{0,0}] \\ \Lambda^3 &= [\lambda^{0,3}] \oplus [\lambda_0^{1,2}] \oplus [\lambda^{0,1}]. \end{aligned}$$

The decomposition of Λ^2 can be identified with

$$\mathfrak{so}(2m) = \mathfrak{u}(m)^\perp \oplus \mathfrak{u}(m)$$

(as $\mathfrak{u}(m)^\perp = [\lambda^{0,2}]$ and $\mathfrak{u}(m) = [\lambda^{1,1}]$).

A priori $\nabla\omega$ lies in $\Lambda^1 \otimes \Lambda^2$, though Salamon proves that $\nabla\omega$ must lie in

$$\begin{aligned} \Lambda^1 \otimes \mathfrak{u}(m)^\perp &= \Lambda^1 \otimes [\lambda^{0,2}] \\ &= [\lambda^{0,1} \otimes \lambda^{0,2}] \oplus [\lambda^{1,2}] \\ &= [V] \oplus [\lambda^{0,3}] \oplus [\lambda_0^{1,2}] \oplus [\lambda^{0,1}] \end{aligned}$$

where $[V]$ is the kernel of the skew-symmetrization $\Lambda^1 \otimes \Lambda^2 \to \Lambda^3$ and the final decomposition is into irreducible representations. We see that $d\omega$ is the component of $\nabla\omega$ lying

$$\Lambda^3 = [\lambda^{0,3}] \oplus [\lambda_0^{1,2}] \oplus [\lambda^{0,1}].$$

The Nihenhuis tensor N takes a $(1,0)$-form α to $(d\alpha)^{0,2}$ of type $(0,2)$, and therefore lies in

$$[\mathrm{Hom}(\lambda^{1,0}, \lambda^{0,2})] = [\lambda^{0,1} \otimes \lambda^{0,2}] = [V] \oplus [\lambda^{0,3}]$$

where we have used the Hermitian metric to identify $(\lambda^{1,0})^*$ and $\lambda^{0,1}$. In fact, one can show that N is the component of $\nabla\omega$ lying in $[V] \oplus [\lambda^{0,3}]$.

It follows that if M is Kähler ($N = 0$ and $d\omega = 0$) then all components of $\nabla\omega$ vanish, as we have seen.

The holonomy reduction of a Kähler manifold can be summarized in the following diagram, also taken from [**S**].

Oriented Riemannian
$\mathrm{SO}(2m),\ g$
$\uparrow$
Kähler
$\mathrm{U}(m)$
$\swarrow \qquad \searrow$

Complex	Symplectic
$\mathrm{GL}(m,\mathbb{C}),\ I$	$\mathrm{Sp}(m,\mathbb{R}),\ \omega$

All the groups appearing are subgroups of $\mathrm{GL}(2m,\mathbb{R})$, and adjacent is the tensor giving the reduction to that group. Note that $\mathrm{Sp}(m,\mathbb{R})$ is the real form of $\mathrm{Sp}(m,\mathbb{C})$, which is the subgroup of $\mathrm{GL}(2m, \mathbb{R})$ preserving the symplectic form ω. Note that the intersection of any two of the 'outer' groups is $\mathrm{U}(m)$, as if any two of g, I, and ω are constant, the third must be too.

Before returning to hyperkähler manifolds, we will prove a result on Kähler manifolds with holonomy contained in $\mathrm{SU}(m)$. This includes hyperkähler manifolds as $\mathrm{Sp}(m) \subset \mathrm{SU}(2m)$.

PROPOSITION 2.8. *A simply-connected Kähler manifold (M, g, I, ω) is Ricci-flat if and only if it has holonomy group* $\mathrm{Hol}(g)$ *contained in* $\mathrm{SU}(m)$. *Moreover, it must have trivial canonical bundle $K = \mathcal{O}_M$.*

PROOF. In chapter one we defined $\mathcal{R}^H$, for a holonomy group H, to be the subspace of $\mathrm{Sym}^2\mathfrak{h}$ satisfying the first Bianchi identity. One can show that

$$\mathcal{R}^H = \mathrm{Sym}^2\mathfrak{h} \cap \ker(a : \Lambda^2 \otimes \Lambda^2 \to \Lambda^4)$$

where a is the skew-symmetrization map. The curvature R of a manifold with holonomy group H must lie in $\mathcal{R}^H$ at each point.

Using representation theory, one obtains the decomposition

$$\mathcal{R}^{\mathrm{U}(m)} = B \oplus \lambda_0^{1,1} \oplus \lambda^{0,0}$$

into irreducible representations of $\mathrm{U}(m)$, where B is some irreducible representation which we will not specify further. The corresponding components of R are the Bochner tensor, the trace-free Ricci curvature, and the scalar curvature. On the other hand, for $\mathrm{SU}(m)$ we find $\mathcal{R}^{\mathrm{SU}(m)} = B$. Therefore if g has holonomy contained in $\mathrm{SU}(m)$, the Ricci curvature must vanish.

To avoid decomposing representations, one can instead argue as follows. The determinant

$$\det : \mathrm{U}(m) \to \mathrm{U}(1) \subset \mathrm{Aut}(\lambda^{m,0})$$

gives a representation of $\mathrm{U}(m)$ on $\lambda^{m,0}$. The frame bundle and connection then induce a connection on the bundle associated to $\lambda^{m,0}$, which is the canonical bundle $K = \Lambda^{m,0}T^*M$. This representation becomes trivial when restricted to the kernel $\ker(\det) = \mathrm{SU}(m)$. Therefore if the holonomy group is contained in $\mathrm{SU}(m)$, the canonical bundle and its induced connection must be trivial. The derivative of the determinant is the trace, and we have a map

$$\lambda^{1,1} \otimes \mathrm{tr} = \lambda^{1,1} \otimes \det' : \lambda^{1,1} \otimes \mathfrak{u}(m) \to \lambda^{1,1} \otimes \mathrm{End}(\lambda^{m,0}) \cong \lambda^{1,1}.$$

The curvature R of a Kähler manifold sits inside $\lambda^{1,1} \otimes \mathfrak{u}(m)$, and the above map takes it to the Ricci curvature Ric. Thus we see that the curvature of the induced connection on K is the Ricci curvature. When $\mathrm{Hol}(g) \subset \mathrm{SU}(m)$ the induced connection is trivial and the Ricci curvature therefore vanishes.

Conversely, if (M, g) is Ricci-flat, then the induced connection on K is flat, and K admits a constant trivializing section Ω (here we use the fact that M is simply-connected). This constant tensor reduces the holonomy from $\mathrm{U}(m)$ to $\mathrm{SU}(m)$. □

REMARK 2.9. Simply knowing that the canonical bundle K is trivial is a numerical condition and not particularly strong, as it tells us nothing about the induced connection on K (which could be non-trivial). Later we will discuss Yau's theorem, which in the compact case asserts the existence of a special metric (roughly, it says that if K is trivial then there exists a Kähler metric on M which induces the trivial connection on K).

REMARK 2.10. The hypothesis that M is simply-connected is important. For example, take a K3 surface S with a fixed-point free involution σ. The quotient $S/\langle\sigma\rangle$ is an Enriques surface, with fundamental group $\mathbb{Z}_2$. Later we will see that S admits a Ricci-flat metric, which by averaging can be made invariant under σ. Thus we get a Ricci-flat metric on the Enriques surface, whose canonical bundle is two-torsion but non-trivial.

We defined a hyperkähler manifold to be a Riemannian manifold (M,g) with constant tensors I, J, K, ω_1, ω_2, and ω_3. From the above considerations, we see that (M,g) is Kähler with respect to each of the complex structure I, J, and K. Moreover, for $(a,b,c) \in \mathbb{R}^3$ with $a^2+b^2+c^2=1$, the combination $aI+bJ+cK$ is also a constant complex structure with constant Hermitian form $a\omega_1+b\omega_2+c\omega_3$, and hence gives another Kähler structure. Thus (M,g) has an S^2 family of Kähler structures, suggesting the name 'hyperkähler'.

We can now prove some equivalent characterizations of hyperkähler manifolds.

PROPOSITION 2.11. *Let (M,g) be a Riemannian manifold with a triple of almost complex structures I, J, and K (satisfying the quaternion relations), such that g is Hermitian with respect to I, J, and K (thus $g(IX,IY)=g(X,Y)$, and so on) with corresponding Hermitian forms ω_1, ω_2, and ω_3. The following are equivalent*

(1) $(M,g,I,J,K,\omega_1,\omega_2,\omega_3)$ *is a hyperkähler manifold,*
(2) ω_1, ω_2, *and* ω_3 *are closed,* $d\omega_1=d\omega_2=d\omega_3=0$,
(3) ω_1, ω_2, *and* ω_3 *are constant,* $\nabla\omega_1=\nabla\omega_2=\nabla\omega_3=0$,
(4) $\mathrm{Hol}(g)\subset\mathrm{Sp}(m)$ *and* I, J, *and* K *are the induced almost complex structures.*

The last statement means that I, J, and K are constant tensors arising from the holonomy reduction.

PROOF. Most of the implications are obvious or follow automatically from what we have already seen. If (1) is true, then (2) and (3) are obvious. The holonomy reduction follows from the existence of constant tensors $(I,J,K,\omega_1,\omega_2,\omega_3)$ on (M,g). If (2) is true, then (3) is obvious. Combined with $\nabla g=0$, we deduce that the almost complex structures are constant, and hence we have both (1) and (4). If (4) is true, then the Hermitian forms must also be constant and we obtain the other three statements. To complete the proof, we will argue that $d\omega_1=d\omega_2=d\omega_3=0$ implies $\nabla\omega_1=\nabla\omega_2=\nabla\omega_3=0$.

The relation $I=JK$ implies an equivalent algebraic relation between ω_1, ω_2, and ω_3, which will be of the form $\omega_1=C(\omega_2\otimes\omega_3)$ where C is some $\mathrm{O}(4m)$-equivariant linear map. We saw above that if a Hermitian form ω is closed then $\nabla\omega$ (at each point) lies in the irreducible representation $[V]$ which is the kernel of the skew-symmetrization $\Lambda^1\otimes\Lambda^2\to\Lambda^3$. Therefore the right hand side of

$$\nabla\omega_1=C(\nabla\omega_2\otimes\omega_3+\omega_2\otimes\nabla\omega_3)$$

lies in $[V\otimes\lambda^{2,0}]$.[1] Decomposing this tensor product into irreducible representations shows that it does not contain $[V]$. Therefore, since the left hand side $\nabla\omega_1$ lies in $[V]$, it must vanish. Similarly $\nabla\omega_2$ and $\nabla\omega_3$ both vanish. □

[1] Is this really true? Note that ω_2 and ω_3 are not of type $(1,1)$ with respect to I, so do $\nabla\omega_2$ and $\nabla\omega_3$ still lie in $[V]$?

REMARK 2.12. Unlike the Kähler case, we do not need to assume integrability of the almost complex structures: this follows automatically from the closure of the three Hermitian forms (there is no almost hyperkähler geometry).

There is a diagram for the holonomy reduction of hyperkähler manifolds[2] similar to the one for Kähler manifolds.

Oriented Riemannian
$\mathrm{SO}(4m),\ g$
$\uparrow$
Hyperkähler
$\mathrm{Sp}(m)$
$\swarrow \qquad\qquad \searrow$

Hypercomplex — $\mathrm{GL}(m,\mathbb{H}),\ I,J,K$

Complex symplectic — $\mathrm{Sp}(2m,\mathbb{C}),\ \omega_1,\omega_2+i\omega_3$

All the groups appearing are subgroups of $\mathrm{GL}(4m,\mathbb{R})$ and the intersection of any two of the 'outer' groups is $\mathrm{Sp}(m)$.

The next result is more or less equivalent to the implication (2)$\Rightarrow$(1) in the last proposition. However, the proof we gives avoids representation theory.

PROPOSITION 2.13. *Let M be a smooth manifold of real-dimension $4m$, with a triple of two-forms ω_1, ω_2, and ω_3 which reduce the structure group to $\mathrm{Sp}(m)$ (that is, for all $p \in M$ the subgroup of $\mathrm{GL}(T_pM) = \mathrm{GL}(4m,\mathbb{R})$ fixing $\omega_1|_p$, $\omega_2|_p$, and $\omega_3|_p$ is $\mathrm{Sp}(m)$). Then M is hyperkähler if and only if ω_1, ω_2, and ω_3 are closed.*

REMARK 2.14. We do not specify a metric on M, but one will be induced from the two-forms.

PROOF. Regard a two-form ω as a section of $\Lambda^2T^*M \subset \mathrm{Hom}(TM,T^*M)$. Then we can define
$$I := -\omega_2^{-1}\omega_3 \in \mathrm{Hom}(TM,TM)$$
and I is an almost complex structure ($I^2=-1$). Thus we get a metric[3] defined by
$$g(X,Y) := \omega_1(X,IY).$$
Since ω_1, ω_2, and ω_3 reduce the structure group to $\mathrm{Sp}(m)$ they satisfy algebraic relations which ensure that we get the same metric by using J or K. Now
$$\begin{aligned}(\omega_2+i\omega_3)(IX,Y) &= \omega_2(JKX,Y)+i\omega_3(-KJX,Y)\\ &= g(-KX,Y)+ig(JX,Y)\\ &= -\omega_3(X,Y)+i\omega_2(X,Y)\\ &= i(\omega_2(X,Y)+i\omega_3(X,Y))\end{aligned}$$

and therefore $\sigma := \omega_2+i\omega_3$ is of type $(2,0)$ with respect to I. It is closed as $d\omega_2 = d\omega_3 = 0$. The algebraic relations on the two-forms also ensure that σ is non-degenerate so that σ^m is a non-vanishing $(2m,0)$-form on M.

[2] This diagram is not quite correct. The subgroup fixing $\omega_2+i\omega_3$ is $\mathrm{Sp}(2m,\mathbb{C})$, and the subgroup fixing all three Hermitian forms is $\mathrm{Sp}(m)$.

[3] I'm not sure how to show this is positive definite.

We will show that I is a complex structure. Let θ be an arbitrary $(1,0)$-form. Then $\theta \wedge \sigma^m$ vanishes, because it is of type $(2m+1,0)$. Therefore

$$d(\theta \wedge \sigma^m) = d\theta \wedge \sigma^m = (d\theta)^{0,2} \wedge \sigma^m$$

also vanishes. This implies $(d\theta)^{0,2}$ vanishes, and hence I is integrable. By the Newlander-Nirenberg theorem, I is a complex structure, and (M, g, I, ω_1) is a Kähler manifold. The same argument applies to J and K, and hence M is hyperkähler. □

Finally, we'd like to point out another difference between Kähler and hyperkähler manifolds. On a Kähler manifold, adding $i\partial\bar{\partial}f$ to ω (where f is a small smooth function) gives a new Kähler metric. The space of Kähler metric is therefore infinite dimensional. On a hyperkähler manifold, if we add $i\partial\bar{\partial}f$ to ω_1 the new metric may no longer be Hermitian with respect to J and K. In other words, there are additional conditions which f must satisfy so that the new metric is still hyperkähler. In fact, the moduli space of hyperkähler metrics is finite dimensional (we will look more closely at the moduli space of hyperkähler structures when we come to studying compact manifolds).

The above comments also illustrate that hyperkähler structures are much more rigid than Kähler structures. In particular, this makes it rather difficult to find examples of hyperkähler metrics. We will study various construction techniques in the following chapters. One guiding principle is to start with a manifold admitting a holomorphic symplectic form, and then try to find a hyperkähler metric on it; we will see various instances of this approach.

CHAPTER 3

Early examples

The earliest examples of hyperkähler metric were discovered by physicists looking for gravitational counterparts to instantons. A (gauge) instanton on M is a self-dual connection on some bundle over M (we will study these in more detail in a later chapter). In four dimensions, Kähler metrics which are Ricci-flat are known as gravitational instantons because the Levi-Civita connection is a self-dual connection on the tangent bundle. If there is some additional symmetry, it is possible to explicitly write down a hyperkähler metric.

3.1. The Eguchi-Hanson metric

Eguchi and Hanson [**EH**] discovered a hyperkähler metric on the four-manifold X whose underlying smooth manifold is that of the cotangent bundle $T^*\mathbb{CP}^1$ to $\mathbb{CP}^1$. Indeed, X is even biholomorphic to $T^*\mathbb{CP}^1$ with respect to one of the complex structures, as we will see. We follow Dancer's description [**D**].

Let r be a radial coordinate and let σ_1, σ_2, and σ_3 be left-invariant one-forms on S^3. We define the metric g by

$$ds^2 = W^{-1}dr^2 + \frac{1}{4}r^2(\sigma_1^2 + \sigma_2^2 + W\sigma_3^2)$$

where $W = 1 - \frac{a^4}{r^4}$ and a is a (non-negative) real parameter. Let $\frac{\partial}{\partial r}$, X_1, X_2, and X_3 be vector fields dual to dr, σ_1, σ_2, and σ_3. We define almost complex structures

$$I: \begin{array}{ccc} \frac{\partial}{\partial r} & \mapsto & \frac{2}{rW}X_3 \\ X_1 & \mapsto & X_2 \end{array}$$

and

$$J: \begin{array}{ccc} \frac{\partial}{\partial r} & \mapsto & \frac{2}{r\sqrt{W}}X_1 \\ X_2 & \mapsto & \frac{1}{\sqrt{W}}X_3 \end{array}.$$

It is straight-forward to check that I and J anti-commute, for example

$$JI\frac{\partial}{\partial r} = J\frac{2}{rW}X_3 = -\frac{2}{r\sqrt{W}}X_2 = -I\frac{2}{r\sqrt{W}}X_1 = IJ\frac{\partial}{\partial r}.$$

One can than show that I and J are covariant constant with respect to the Levi-Civita connection g, which is therefore hyperkähler. Alternately, one can show that g is Ricci-flat and Kähler with respect to I (which is integrable), as this also shows that the holonomy reduces to $\mathrm{SU}(2) = \mathrm{Sp}(1)$.

If $a = 0$ we simply get the Euclidean metric on $\mathbb{R}^4$. For positive values of a, we let $r \in [a, \infty)$. There is an apparent singularity at $r = a$, but a change of coordinates shows that the metric is actually smooth here; indeed g is a smooth complete metric. There is an action of $\mathrm{SU}(2) = S^3$ on the underlying four-manifold X. The orbits are $S^3/\mathbb{Z}_2 = \mathbb{RP}^3$ for $r > a$ and S^2 for $r = a$. Actually (X, I) is biholomorphic to the total space of the cotangent bundle $T^*\mathbb{CP}^1$, where the zero section $\mathbb{CP}^1$ is the S^2 at $r = a$.

The metric can also be defined by giving its Kähler potential (as in Joyce's book [**J**]). We begin with $\mathbb{C}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by multiplication by -1 with an isolated singularity at the origin. This orbifold has a quaternionic structure given by identifying $\mathbb{C}^2$ with $\mathbb{H}$, but to resolve the singularity we fix the complex structure I and blow-up the origin to obtain $\pi : X \to \mathbb{C}^2/\mathbb{Z}_2$. The resolution X is $T^*\mathbb{CP}^1$, where the exceptional divisor $E := \pi^{-1}(0)$ is a rational curve of self-intersection -2 which is identified with the zero section of the cotangent bundle. Define a function $f : X\backslash E \to \mathbb{R}$ by

$$f(\pm(z_1, z_2)) = \sqrt{R^4+1} + 2\log R - \log(\sqrt{R^4+1} + 1)$$

where $R = (|z_1|^2 + |z_2|^2)^{\frac{1}{2}}$. This function extends smoothly to X and gives a Kähler potential for $\omega_1 = i\partial\bar{\partial} f$. The holomorphic symplectic form $dz_1 \wedge dz_2$ on $\mathbb{C}^2$ is invariant under $\mathbb{Z}_2$, so gives a holomorphic two-form on $\mathbb{C}^2/\mathbb{Z}_2$ which can be pulled back to X to give

$$\sigma = \omega_2 + i\omega_3 = \pi^*(dz_1 \wedge dz_2).$$

Moreover, one can show that this defines a non-degenerate two-form on X which is constant.

The Eguchi-Hanson metric is an example of an ALE space (asymptotically locally Euclidean). This means that g approaches the Euclidean metric on $\mathbb{C}^2/\mathbb{Z}_2$ to order R^{-4} for large R (with similar asymptotic behaviour for the derivatives of the metric), as can be seen by observing that

$$\omega_1 = i\partial\bar{\partial}(R^2) + O(R^{-4})$$

and $i\partial\bar{\partial}(R^2)$ is just the Kähler form of the Euclidean metric. Later we will see other examples of ALE spaces, which arise by resolving $\mathbb{C}^2/\Gamma$ where Γ is a finite subgroup of $\mathrm{SU}(2)$.

We have concentrated on a specific complex structure on X, with respect to which X is a resolution of $\mathbb{C}^2/\mathbb{Z}_2$. There is an embedding into $\mathbb{C}^3$ given by

$$\pm(z_1, z_2) \mapsto (z_1^2, z_2^2, z_1 z_2)$$

which realizes $\mathbb{C}^2/\mathbb{Z}_2$ as the singular quadric $\{xy - z^2 = 0\} \subset \mathbb{C}^3$. In fact, with respect to the other complex structures X is biholomorphic to the smooth quadric $\{xy - z^2 = 1\}$; in particular, it is an affine variety (it can be described as the zero locus of a set of equations in $\mathbb{C}^3$). By contrast $(X, I) \cong T^*\mathbb{CP}^1$ is not affine as the zero section $\mathbb{CP}^1$ clearly cannot be embedded in any $\mathbb{C}^N$. We will see that this kind of behaviour is quite common when we vary the complex structure (in the S^2 of complex structures compatible with the metric) on a hyperkähler manifold.

3.2. The Gibbons-Hawking ansatz

Gibbons and Hawking [**GH**] found a general ansatz which provides numerous examples of four-dimensional hyperkähler manifolds. Let (x, y, z) be coordinates on $\mathbb{R}^3$ and θ a coordinate on S^1. The metric g is defined by

$$ds^2 = V(dx^2 + dy^2 + dz^2) + V^{-1}(d\theta + \alpha)^2$$

where $V := V(x, y, z)$ is a function and $\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$ is a one-form on $\mathbb{R}^3$. In order for g to be Ricci-flat and give an example of a hyperkähler metric, we require $*d\alpha = dV$ on $\mathbb{R}^3$ where $*$ is the Hodge star operator (with respect to the Euclidean metric on $\mathbb{R}^3$). Regarding α as a vector field, we can write this more classically as $\text{curl}\vec{\alpha} = \text{grad}V$.

The Lie derivative of g with respect to $\frac{\partial}{\partial\theta}$ vanishes, and thus $\frac{\partial}{\partial\theta}$ is a Killing vector field (ie. it generates an isometry). It follows that the underlying space X is a principal S^1-bundle over $\mathbb{R}^3$, the quotient metric on $\mathbb{R}^3$ is $V(dx^2 + dy^2 + dz^2)$, the connection on the S^1-bundle is $d\theta + \alpha$, and the metric on each fibre is $V^{-1}d\theta^2$. In particular, the circumference of the fibre above (x, y, z) is $V(x, y, z)^{-1}$, so when $V \to \infty$ the fibre collapses to a point.

We will consider some specific choices of V:

(1) $V = \frac{1}{r}$ where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ gives the Euclidean metric on $\mathbb{R}^4$,
(2) $V = \frac{1}{r} + c$ where $c > 0$ gives the Taub-NUT metric on $\mathbb{R}^4$,
(3) $V = \sum_{i=1}^n \frac{1}{|\vec{x}-\vec{a}_i|}$ where $\{\vec{a}_i\}$ is a collection of distinct points in $\mathbb{R}^3$ gives the multi-instanton metrics,
(4) $V = (\sum_{i=1}^n \frac{1}{|\vec{x}-\vec{a}_i|}) + c$ gives the multi-Taub-NUT metrics.

The Taub-NUT metric was first discovered in 1951. As with the Euclidean metric, if we first choose an identification $\mathbb{R}^4 \cong \mathbb{C}^2$, then the S^1 action is given by $(z_1, z_2) \mapsto (e^{i\theta}z_1, e^{-i\theta}z_2)$. By choosing S^1-invariant functions

$$x = |z_1|^2 - |z_2|^2 \qquad \text{and} \qquad y + iz = 2z_1z_2$$

on $\mathbb{C}^2$, we obtain local coordinates (x, y, z) on the quotient $\mathbb{R}^3$. This gives $\mathbb{R}^4$ the structure of an S^1-bundle over $\mathbb{R}^3$, with the fibre over the origin collapsing to a point. Note that

$$r = (x^2 + y^2 + z^2)^{\frac{1}{2}} = |z_1|^2 + |z_2|^2.$$

For the Euclidean metric the circumference $V^{-1} = r$ of the S^1 fibre grows with r, whereas for the Taub-NUT metric $V^{-1} = \frac{1}{r^{-1}+c}$ tends to the finite value $\frac{1}{c}$ as $r \to \infty$. Thus the Taub-NUT metric is not ALE, but it is ALF (asymptotically locally flat). This means that the metric is asymptotic to $\mathbb{R}^3 \times S^1$ at infinity, a property which is also shared by the multi-Taub-NUT metrics.

Let us look more closely at the multi-instanton metrics. When $\vec{x} = \vec{a}_i$ the circle fibre collapses to a point, so above a segment connecting $\vec{a}_i$ to $\vec{a}_j$ we get a two-sphere in X, with the two collapsed fibres as north and south poles[1]. The topology of X comes from these two-spheres. For example, when $n = 2$ there is exactly one two-sphere and we recover the Eguchi-Hanson metric. More generally, when $\vec{a}_i$ are colinear we obtain a chain of $n - 1$ two-spheres; this is the ALE space of type A_{n-1} which we will see more of later.

To see why the multi-instanton metrics are ALE, we will show that they are asymptotic to the Euclidean metric on $\mathbb{C}^2/\mathbb{Z}_n$. In this quotient the generator $\zeta = e^{\frac{2\pi i}{n}}$ of $\mathbb{Z}_n$ acts by $(z_1, z_2) \mapsto (\zeta z_1, \zeta^{-1}z_2)$. In other words, we can think of $\mathbb{Z}_n$ as a subgroup of S^1, which acts on $\mathbb{C}^2$ as described above. The Euclidean metric on $\mathbb{C}^2/\mathbb{Z}_n$ is therefore obtained by dividing the S^1 coordinate θ by n. Now for large r

$$V = \sum_{i=1}^n \frac{1}{|\vec{x} - \vec{a}_i|} \sim \frac{n}{r}$$

[1]Include a picture for this.

and therefore

$$\begin{aligned} ds^2 &\sim \frac{n}{r}(dx^2+dy^2+dz^2)+\frac{r}{n}(d\theta+\alpha)^2 \\ &= n\left(\frac{1}{r}(dx^2+dy^2+dz^2)+r\left(d\left(\frac{\theta}{n}\right)+\frac{\alpha}{n}\right)^2\right) \end{aligned}$$

which is the Euclidean metric on $\mathbb{C}^2/\mathbb{Z}_n$ (up to an overall factor).

Note that taking the $\mathbb{Z}_n$-orbit of (z_1, z_2) to $(z_1^n, z_2^n, z_1 z_2) \in \mathbb{C}^3$ embeds $\mathbb{C}^2/\mathbb{Z}_n$ as the singular affine variety $\{xy - z^n = 0\} \subset \mathbb{C}^3$. The singularity can be resolved by successive blow-ups, resulting in an exceptional divisor which consists of a chain of $n-1$ rational curves. Choose $\vec{a}_i$ to be colinear. Then with respect to one of its complex structures, X is biholomorphic to this resolution of $\mathbb{C}^2/\mathbb{Z}_n$, whereas for the other complex structures X is biholomorphic to the smooth affine variety $\{xy - z^n = 1\} \subset \mathbb{C}^3$.

3.3. The Calabi metric

Another generalization of the Eguchi-Hanson metric is the Calabi metric on the total space of the cotangent bundle $T^*\mathbb{CP}^n$ [**C**]. More generally, let us first consider the total space of the cotangent bundle $\pi : T * M \to M$ of an arbitrary Kähler manifold M. It admits a canonical holomorphic symplectic form defined in the following way. Let $(z^1, \ldots, z^n)$ be local coordinates on M, so that a point of T^*M looks like $\sum_\alpha w_\alpha dz^\alpha$. The tautological one-form on T^*M is given by $\eta = \sum_\alpha w_\alpha \pi^* dz^\alpha$ and the holomorphic symplectic form is

$$\sigma := d\eta = \sum_\alpha dw_\alpha \pi^* dz^\alpha.$$

An obvious question is whether there exists a Kähler metric on $T * M$ such that σ is covariant constant with respect to the Levi-Civita connection, as this metric would then be hyperkähler. This is the case when M is complex projective space.

THEOREM 3.1 (Calabi [**C**]). *There exists a complete hyperkähler metric on the total space of $T^*\mathbb{CP}^n$.*

PROOF. The Kähler potential of the hyperkähler metric must satisfy certain equations, and Calabi's proof involves explicitly solving these equations. We will not present the details as the metric allows a much simpler description via the hyperkähler quotient construction, that we will see in a later chapter. Moreover, the quotient construction also yields hyperkähler metrics on the cotangent bundles of Grassmannians (cf. Lindström and Roček [**LR**][2]) and arbitrary generalized flag manifolds. □

REMARK 3.2. Calabi also showed that if M is Kähler-Einstein with scalar curvature $t \neq 0$, then there exists a Ricci-flat Kähler metric on a neighbourhood of the zero section of the canonical line bundle $K_M = \Omega^{n,0} \to M$. Once again, this was achieved by explicitly finding the Kähler potential. If $t > 0$ then the metric extends globally to all of K_M. For example, when $n = 1$ and M is $\mathbb{CP}^1$ this is the Eguchi-Hanson metric. If M is $\mathbb{CP}^n$ we get an ALE metric, asymptotic to the

[2] I need to check this reference, as it is four years earlier than the paper introducing hyperkähler quotients!

Euclidean metric on $\mathbb{C}^{n+1}/\mathbb{Z}_{n+1}$. Of course, for $n > 1$ these metrics are Calabi-Yau but not hyperkähler.

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