MAT 569 : Differential Geometry (II)

Chapter 4 - Curvature

Denote by $\chi(M)$ the space of vector fields on the Riemannian manifold M, and by ∇ the Levi-Civita connection (the unique torsion-free connection compatible with the metric).

Definition *The* curvature *R of M associates to each pair X, $Y \in \chi(M)$ a map*

$$R(X,Y) : \chi(M) \to \chi(M)$$

$$Z \mapsto \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.^\dagger$$

Exercise: 1. Show R is bilinear in X and Y, and that $R(X,Y)$ is linear in Z for fixed X and Y.
2. Define $(X,Y,Z,T) := \langle R(X,Y)Z, T\rangle$. Show that

1. $(Y,X,Z,T) = -(X,Y,Z,T)$,
2. $(X,Y,T,Z) = -(X,Y,Z,T)$,
3. $(X,Y,Z,T) + (Y,Z,X,T) + (Z,X,Y,T) = 0$ (the Bianchi identity),
4. $(Z,T,X,Y) = (X,Y,Z,T)$.

Denote $(|x|^2|y|^2 - \langle x,y\rangle^2)^{1/2}$ by $|x \wedge y|$.

Definition *The* sectional curvature *of a two-dimensional subspace $\sigma \subset T_pM$ is defined as*

$$K(\sigma) := K(x,y) := \frac{(x,y,x,y)}{|x \wedge y|^2}$$

where x and $y \in T_pM$ are a pair of basis vectors for σ.

Exercise: 3. Show that the definition of $K(\sigma)$ does not depend on the choice of basis vectors x and y.
4. For fixed p, prove that knowledge of $K(\sigma)$ for *all* two-dimensional subspaces $\sigma \subset T_pM$ uniquely determines the curvature R at p.

†Note that the do Carmo uses the opposite sign, which is less standard.

Lemma 1 *Define a trilinear map* $R' : T_pM \times T_pM \times T_pM \to T_pM$ *by*

$$\langle R'(X,Y,W), Z\rangle := \langle X,W\rangle\langle Y,Z\rangle - \langle Y,W\rangle\langle X,Z\rangle$$

for all X, Y, W, $Z \in T_pM$. *Then* M *has constant sectional curvature* K *iff* $R = KR'$.

Proof It is enough to verify that R' satisfies the symmetries of a Riemannian curvature tensor (ie. those appearing in Exercise 2), and has constant sectional curvature 1. The lemma then follows from Exercise 4. □

Remark Choosing an orthonormal basis for T_pM, we can write in coordinates

$$R_{ijkl} = K(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

Choose a unit vector $x = z_n$ in T_pM. Complete it to an orthonormal basis $\{z_1, \ldots, z_{n-1}, z_n\}$ for T_pM.

Definition *The* Ricci curvature *at* p *in the direction* x *is*

$$\mathrm{Ric_p(x)} := \frac{1}{\mathrm{n-1}}\sum_{\mathrm{i=1}}^{\mathrm{n-1}}\langle \mathrm{R(x,z_i)x, z_i}\rangle$$

and the scalar curvature *at* p *is*

$$K(p) := \frac{1}{n}\sum_{j=1}^{n}\mathrm{Ric_p(z_j)}.$$

Exercise: 5. Show that these definitions do not depend on the vectors chosen to complete the orthonormal basis $\{z_1, \ldots, z_{n-1}, z_n\}$.