MAT 569 Differential Geometry II : Exercise Sheet Four

1. A manifold M^n is *Einstein* if $\mathrm{Ric} = \lambda g$ for some function $\lambda : M \to \mathbb{R}$. Prove that if $n \geq 3$ and M is connected then λ must be a constant. Prove that if $n = 3$ then M^3 has constant sectional curvature.

2. Put the metric
$$g_{ij} = \frac{\delta_{ij}}{F^2}$$
on a neighbourhood in $\mathbb{R}^n$, $n \geq 3$, where F is a non-zero function on $\mathbb{R}^n$.

   (i) Prove that the metric has constant curvature K iff
$$F = G_1(x_1) + \ldots + G_n(x_n)$$
where each term is a quadratic
$$G_i(x_i) = ax_i^2 + b_i x_i + c_i$$
and
$$\sum_{i=1}^{n} (4ac_i - b_i^2) = K.$$

   (ii) Putting $a = K/4$, $b_i = 0$, and $c_i = 1/n$ we obtain the metric
$$g_{ij} = \frac{\delta_{ij}}{(1 + \frac{K}{4}\sum x_i^2)^2}.$$
Show that for $K > 0$ this gives a metric on all of $\mathbb{R}^n$ which is *not* complete.

3. Let p and q be relatively prime integers, $q > 2$. Consider the sphere
$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2 = \mathbb{R}^4$$
and define $h : S^3 \to S^3$ by
$$h(z_1, z_2) = (e^{\frac{2\pi i}{q}} z_1, e^{\frac{2\pi i p}{q}} z_2).$$

   (i) Show that h generates a group (isomorphic to $\mathbb{Z}_q$) of isometries of S^3 which acts in a totally discontinuous manner. The quotient $S^3/\mathbb{Z}_q$ is called a *Lens space*.

   (ii) Show that every geodesic of $S^3/\mathbb{Z}_q$ is closed, but they can be of different lengths. What is the diameter of $S^3/\mathbb{Z}_q$?

4. Let
$$S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : |z_0|^2 + \ldots + |z_n|^2 = 1\} \subset \mathbb{C}^{n+1} - \{0\} = \mathbb{R}^{2n+2} - \{0\}.$$

   (i) Show that the equivalence relation $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$, $\lambda \in \mathbb{C}^*$ on $\mathbb{C}^{n+1} - \{0\}$ induces the equivalence relation $(z_0, \ldots, z_n) \sim (e^{i\theta} z_0, \ldots, e^{i\theta} z_n)$ on S^{2n+1}. Thus we obtain a smooth fibration
$$f : S^{2n+1} \to \mathbb{CP}^n$$

with S^1 fibres, known as the *Hopf fibration.*

(ii) Show that if $\mathbb{CP}^n$ has the Fubini-Study metric than f is a *Riemannian submersion*, i.e. f is a submersion in the differential topological sense (its differential df has maximal rank at all points), and df preserves the lengths of vector orthogonal to the fibres.

5. Let X and Y be orthonormal tangent vectors to $\mathbb{CP}^n$. Using the Riemannian submersion from the previous question, lift them *horizontally* to $\bar{X}$ and $\bar{Y}$, tangent vectors to S^{2n+1} (the metric on S^{2n+1} splits each tangent space into a vertical part, tangent to the fibre, and a horizontal part). Multiplication by i preserves the horizontal part of the tangent bundle to the sphere, thought of as a subbundle of $T\mathbb{C}^{n+1}|_{S^{2n+1}}$. So let ϕ be the angle between $\bar{X}$ and $i\bar{Y}$, i.e.
$$\cos\phi = g_{S^{2n+1}}(\bar{X}, i\bar{Y}).$$
Show that the sectional curvature of $\mathbb{CP}^n$ for the plane σ generated by X and Y is
$$K(\sigma) = 1 + 3\cos^2\phi.$$
In particular, projective space is quarter-pinched $1 \leq K \leq 4$.

6. Let M be a complete simply connected Riemannian manifold with non-positive sectional curvature. Let $\gamma : \mathbb{R} \to M$ be a geodesic parametrized by arc length, and let $p \in M \backslash \gamma(\mathbb{R})$. Define $d(s) := d(p, \gamma(s))$ and let $\sigma_s : [0, d(s)] \to M$ be the minimizing geodesic from p to $\gamma(s)$.

   (i) For the variation $h(s,t) = \sigma_s(t)$ show that
$$\frac{1}{2}E'(s) = \langle \gamma'(s), \sigma_s'(d(s)) \rangle$$
and
$$\frac{1}{2}E''(s) > 0.$$
Hence d must have a unique critical point s_0 which is a minimum. Moreover
$$\langle \gamma'(s_0), \sigma_s'(d(s_0)) \rangle = 0$$
which means that the shortest curve joining p to γ is perpendicular to γ.

   (ii) Show by examples that simple connectivity and non-positive curvature are necessary hypotheses.

7. Let M be a complete Riemannian manifold with $K \leq K_0$, where K_0 is a positive constant. Suppose that p and q can be joined by two distinct geodesics γ_0 and γ_1, with γ_0 shorter that γ_1. Assume we have a homotopy (continuous family of curves) γ_t, $t \in [0,1]$, between these two geodesics. Prove that some curve γ_{t_0} must be "long", i.e.
$$\ell(\gamma_0) + \ell(\gamma_{t_0}) \geq \frac{2\pi}{K_0}.$$
[This is known as Klingenberg's Lemma. See page 236 of do Carmo's book for a hint.]

8. Use Klingenberg's Lemma to prove Hadamard's Theorem. [Hint: You need to show that any two points are joined by a *unique* geodesic, for then the exponential map must be injective.]

9. A weaker version of Bonnet-Myers' Theorem assumes that M is complete and has *sectional* curvature (instead of Ricci curvature) bounded below, $K \geq 1/r^2 > 0$, to conclude that M is compact and diam $\leq \pi r$ as before. Prove this result using the Rauch Comparison Theorem and the Jacobi Theorem (the latter is the corollary that immediately followed the Morse Index Theorem).

10. (i) Suppose M has Ric $\geq k$ and for some point $p \in M$

$$\mathrm{vol} B(p,r) = v(n,k,r)$$

where $v(n,k,r)$ is the volume of a ball of radius r in an n-dimensional space of constant curvature k. Show that M has constant curvature k on $B(p,r)$.

(ii) Show that a complete manifold M with Ric ≥ 0 and

$$\lim_{r\to\infty} \frac{\mathrm{vol} B(p,r)}{v(n,0,r)} = 1$$

for some $p \in M$, must be isometric to Euclidean space.

11. Prove the converse of Toponogov's Theorem. In other words, if for some k the conclusion of Toponogov's Theorem holds when hinges in M are compared to hinges in a two-dimensional space of constant curvature k, then M has sectional curvature $K \geq k$.

12. Show that for arbitrary vector fields X and Y, the Lie derivative satisfies

$$L_X \circ L_Y - L_Y \circ L_X = L_{[X,Y]}.$$

13. Suppose we have a (local) frame of Killing fields $X_1, \ldots, X_n$ on a Riemannian manifold M. Show that the structure constants c_{ij}^k defined by

$$[X_i, X_j] = c_{ij}^k X_k$$

are constant. [Thus a Killing frame is a finite-dimensional Lie algebra.]

14. Let X be a Killing field on a closed manifold M. Assume that α is a harmonic form.

(i) Show that $L_X \alpha = 0$. [Hint: Show it is harmonic.]

(ii) Show that $i_X \alpha$ is closed, but not necessarily harmonic.

15. Let M^{2n} be a Kähler manifold with Kähler form ω. Show that $\omega^k = \omega \wedge \cdots \wedge \omega$ is closed but not exact, for $k = 1, \ldots, n$. Thus none of the even cohomology groups can vanish. [Hint: Show that ω^n is proportional to the volume form.]

16. Show that a compact manifold with irreducible holonomy and Ric ≥ 0 must have finite fundamental group.

Presentation Topics

1. Describe the flat compact manifolds in dimension three. These are space forms, so must be quotients of $\mathbb{R}^3$. [See pages 231- in "Three-Dimensional Geometry and Topology", William Thurston. He calls these Euclidean manifolds.]

2. Two other interesting space forms are the Poincaré dodecahedral space and the Seifert-Weber dodecahedral space, in positive and negative curvature respectively. Describe these spaces. [See pages 34 and 36 of Thurston's book. The picture on the cover, and on the Perelman posters, is the tiling of $\mathbb{H}^3$ coming from the Seifert-Weber space.]

3. Describe focal points and focal sets of submanifolds. [Page 230 of do Carmo's book.]

4. Describe the basic idea behind Hamilton's proof that any compact simply connected three-manifold with positive Ricci curvature is homeomorphic to S^3. There's lots of analysis, but you can ignore that. [R. Hamilton, "Three-manifolds with positive Ricci curvature", Jour. Diff. Geom, **17** (1982), 255-306.]

5. Describe some of the Cheeger-Gromov compactness results. Here compactness refers to the 'space' of all Riemannian manifolds, which can be topologized in various ways. [See chapter 10 of "Riemannian Geometry", Peter Petersen.]

6. Describe spinor bundles and the Dirac operator. It would suffice to work in dimension four. If possible, describe Weitzenböck's formula. [See chapter 7 and appendix C of Petersen's book.]

7. Describe the generalized Gauss-Bonnet Theorem in dimension greater than two. [See chapter 4 of Petersen's book.]

8. Describe characteristic numbers (Chern or Pontrjagin numbers) and their relation to curvature. [See exercise 23 on page 205 of Petersen's book.]

9. Prove that $\mathbb{CP}^2$ is not even locally a hypersurface in $\mathbb{R}^5$. [See section 4.2 of Petersen's book. You will need to understand the curvature *operator* and why it is *not* strictly positive for $\mathbb{CP}^2$.]

10. Generalize Preissmann's Theorem to show that any *solvable* subgroup must be cyclic.

11. Find out what Yau's Theorem (the Calabi Conjecture) states and what it has to do with the existence of Ricci-flat metric on, for example, hyperkähler or Calabi-Yau manifolds (holonomy group $\mathrm{S}p(m)$ or $\mathrm{S}U(m)$ respectively). [For example, see page 98 of "Compact manifolds with special holonomy", Dominic Joyce.]

12. Describe the proof of Hodge's Theorem. You needn't cover all the analysis. [See "Principles of algebraic geometry", Griffiths and Harris.]