MAT 569 Differential Geometry II : Exercise Sheet Three

1. Let γ be a closed geodesic in an even-dimensional Riemannian manifold M, i.e. γ is an immersion of S^1 which satisfies the geodesic equation at every point. If M has positive curvature, prove that γ is homotopic to a 'nearby' curve of strictly shorter length.

2. Let M be a complete Riemannian manifold with non-positive sectional curvature. Prove that
$$|(d\exp_p)_v(w)| \geq |w|$$
for all $p \in M$, $v \in T_pM$, and $w \in T_v(T_pM)$.

3. A *line* in a complete Riemannian manifold M is a geodesic
$$\gamma : \mathbb{R} \to M$$
which minimizes the length between any two of its points.

   (i) Prove that if M has strictly positive sectional curvature, then it cannot contain a line.

   (ii) Find an example of a manifold with non-negative sectional curvature which contains a line.

4. If N is a totally geodesic submanifold of K, and K is a totally geodesic submanifold of M, show that N is a totally geodesic submanifold of M.

5. (i) Show that the catenoid
$$f(s,t) = (\cosh s \cos t, \cosh s \sin s, s)$$
is a minimal surface in $\mathbb{R}^3$ by explicitly calculating the second fundamental form S and verifying that it has trace zero. What is the Gaussian curvature?

   (ii) Find all surfaces of revolution in $\mathbb{R}^3$ that are minimal.

6. (i) Show that the helicoid
$$f(s,t) = (t \cos s, t \sin s, s)$$
is a minimal surface in $\mathbb{R}^3$ by calculating the induced metric g and then verifying that the coordinate functions of the surface are harmonic with respect to the Laplace-Beltrami operator
$$\Delta h := -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} (\sqrt{g} g^{\alpha\beta} \frac{\partial h}{\partial x^\beta}).$$

   (ii) Now explicitly calculate the second fundamental form of the helicoid and verify that it has trace zero. What is the Gaussian curvature?

7. Repeat the previous problem for Enneper's surface

$$f(s,t) = (\frac{s}{2} - \frac{s^3}{6} + \frac{st^2}{2}, -\frac{t}{2} + \frac{t^3}{6} - \frac{s^2t}{2}, \frac{s^2}{2} - \frac{t^2}{2}).$$

8. Consider the map $x : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$x(\theta, \phi) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\phi, \sin\phi)$$

for $(\theta, \phi) \in \mathbb{R}^2$. This gives an immersion of $\mathbb{R}^2$ into the unit sphere $S^3 \subset \mathbb{R}^4$. Show that the image is the *flat* torus T^2 in the induced metric.

9. In the previous problem, show that T^2 is a minimal submanifold of S^3. Is T^2 also a minimal submanifold of $\mathbb{R}^4$?

10. Suppose we are given a complete metric on $\mathbb{R}^2$ (not necessarily the Euclidean metric). Let $K(x,y)$ be the Gaussian curvature at the point $(x,y) \in \mathbb{R}^2$. Prove that

$$\lim_{r\to\infty} (\inf_{x^2+y^2 \geq r^2} K(x,y)) \leq 0.$$