MAT 569 Differential Geometry II : Exercise Sheet One

1. A function $g : \mathbb{R} \to \mathbb{R}$ taking t to $yt + x$, where x and $y \in \mathbb{R}$, $y > 0$, is called a *proper affine function*. The set of all such functions forms a Lie group G under composition. As a smooth manifold G is the upper half-plane $\{(x, y) \in \mathbb{R}^2 | y > 0\}$ with smooth structure induced from $\mathbb{R}^2$. However, as a Lie group we can give G a left-invariant metric g (given by $dx^2 + dy^2$ at the identity element $e = (0, 1)$).

   (i) Prove that g is the hyperbolic metric
   $$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

   (ii) Writing $z = x + iy$, prove that
   $$z \mapsto \frac{az + b}{cz + d}$$
   is an isometry, where a, b, c, and $d \in \mathbb{R}$ satisfy $ad - bc = 1$.

2. Show that if M is locally isometric to N, then N need not be locally isometric to M.

3. Let X and Y be smooth vector fields on a Riemannian manifold M, and let $c : [0, 1] \to M$ be a flow line of the vector field X starting at the point $p = c(0)$. Let $P_{c,0,t} : T_pM \to T_{c(t)}M$ be parallel transport along c. Prove that the Riemannian connection of M is given by
   $$(\nabla_X Y)(p) = \frac{d}{dt}(P_{c,0,t}^{-1}(Y(c(t))))|_{t=0}.$$

4. If M^m is a smooth submanifold of a Riemannian N^n $(m < n)$ it inherits a Riemannian metric; then M has a natural connection, the Levi-Civita connection ∇ of this induced metric. On the other hand, given vector fields X and Y on M, we can extend them (at least locally) to vector fields X' and Y' on N. Let ∇' be the Levi-Civita connection on N, and consider
   $$(\nabla'_{X'} Y')(p) \in T_pN$$
   where $p \in M$. Prove that the component of the above expression which is tangent to M is precisely $(\nabla_X Y)(p)$. [So the connection of the induced metric is the same as the induced connection.]

5. Let $M^2 \subset \mathbb{R}^3$ be a surface in $\mathbb{R}^3$ with the induced Riemannian metric. Let $c : [0, 1] \to M$ be a smooth curve in M and let V be a smooth vector field *in* M along c (thus V is always tangent to $M \subset \mathbb{R}^3$). We can regard V as a function $V : [0, 1] \to \mathbb{R}^3$ with $V(t) \in T_{c(t)}M \subset T_{c(t)}\mathbb{R}^3 \cong \mathbb{R}^3$.

   (i) Prove that V is parallel iff $\frac{dV}{dt}$ is normal to M, where $\frac{dV}{dt}$ is the usual derivative of a vector-valued function $V : [0, 1] \to \mathbb{R}^3$.

   (ii) For the unit sphere $S^2 \subset \mathbb{R}^3$, show that the velocity vector field along a great circle is a parallel vector field.

   (iii) In Euclidean space, parallel transport between two points does not depend on the path taken. Prove that this is not always true on a general Riemannian manifold.

6. Let $U := (u_0, u_1) \times (v_0, v_1) \subset \mathbb{R}^2$, and let f and h be smooth functions with $(f')^2 + (h')^2$ and f always non-zero.

(i) Show that $\psi : U \to \mathbb{R}^3$ taking (u, v) to $(f(v)\cos u, f(v)\sin u, h(v))$ is an immersion. The image $\psi(U)$ is called a *surface of revolution*.

(ii) Show that the induced metric in the coordinates (u, v) is given by

$$g_{11} = f^2, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = (f')^2 + (h')^2.$$

(iii) Show that the equations of a geodesic γ are

$$\frac{d^2u}{dt^2} + \frac{2f'}{f}\frac{du}{dt}\frac{dv}{dt} = 0,$$

$$\frac{d^2v}{dt^2} - \frac{ff'}{(f')^2 + (h')^2}(\frac{du}{dt})^2 + \frac{f'f'' + h'h''}{(f')^2 + (h')^2}(\frac{dv}{dt})^2 = 0.$$

(iv) Deduce the following geometric meaning of the above equations (except for *meridians* and *parallels*, i.e. when u and v respectively are constant): The second equation says that the "energy" $|\gamma'(t)|^2$ is constant along γ. The first equation says that if $\beta(t) < \pi$ is the angle that γ makes with a parallel (of radius r) at $\gamma(t)$, then $r\cos\beta$ is constant. [The latter is known as Clairaut's relation.]

(v) In the case that $U = (-\epsilon, 2\pi + \epsilon) \times (0, \infty)$, $f(v) = v$, and $h(v) = v^2$ we get a paraboloid. Let γ be a geodesic of the paraboloid which is not a meridian. Use Clairaut's relation to show that γ intersects itself an infinite number of times.

7. Let X be a smooth vector field on a Riemannian manifold M, let U be a neighbourhood of the point $p \in M$, and let $\psi : (-\epsilon, \epsilon) \times U \to M$ be the flow generated by X, i.e. for each $q \in U$, $t \mapsto \psi(t, q)$ is the flow line of X through q. Then for fixed $t_0 \in (-\epsilon, \epsilon)$ $\psi(t_0, -) : U \to M$ is a local diffeomorphism; if it is an isometry then we call X a *Killing field*. Prove the following:

(i) A vector field X on Euclidean space may be seen as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$; we say X is linear if this map is linear. A linear vector field is a Killing field iff the corresponding linear map is given by an anti-symmetric matrix.

(ii) If a Killing field X on M vanishes at $p \in M$, and vanishes nowhere else inside a normal neighbourhood U of p, then X is tangent to the geodesic spheres centred at p and contained in U.

(iii) X is a Killing field iff $\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0$ for all Y and $Z \in \chi(M)$. [This is known as the *Killing equation*.]

(iv) Let X be a Killing field on M such that at some point $q \in M$, $X(q) = 0$ and $\nabla_Y X(q) = 0$ for all $Y_q \in T_q M$. Then X is identically zero.

8. Let $p \in M$. Show that there exists a neighbourhood U of p and n vector fields $E_1, \ldots, E_n$ on U which are orthonormal at each point of U and such that $\nabla_{E_i} E_j(p) = 0$ at p. [Such a family of vector fields is called a *geodesic frame*.]