MAT 566 Differential Topology : Exercise Sheet Four

1. Let M^k be a smooth compact manifold. Prove that M can be embedded in Euclidean space $\mathbb{R}^N$ for some large N without using any of the theorems from class, except for the existence of good atlases and bump functions. [Hint: Choose a finite good atlas $\{(h_i, V_i)\}_{i=1}^s$ and a bump function ψ for the unit ball, supported in the ball of radius two. Show that the map $g : M \to \mathbb{R}^{s(k+1)}$

$$p \mapsto (\psi \circ h_1(p), \psi \circ h_1(p)h_1(p), \ldots, \psi \circ h_s(p), \psi \circ h_s(p)h_s(p))$$

is an embedding.]

[Together with problem 9 on exercise sheet 3, this shows the Whitney Embedding Theorem for compact manifolds: a manifold of dimension k can be embedded in $\mathbb{R}^{2k+1}$. The proof here is much simpler than the one in class, though it only works for compact manifolds.]

2. Let $A \subset M$ be closed, U an open neighbourhood of A, and f a smooth map from U into $\mathbb{R}^n$. Show that there exists a smooth map $g : M \to \mathbb{R}^n$ that agrees with f on A, ie. $g|_A = f|_A$.

3. Construct an injective smooth map $f : S^1 \to \mathbb{R}^2$ whose image is the square

$$\{x \in \mathbb{R}^2 | \max\{|x_1|, |x_2|\} = 1\}.$$

Can f be an immersion?

4. (i) Let M^n be a connected non-compact smooth manifold. Show that there exists a sequence of open subsets $V_i \subset M$ so that

   - each V_i is homeomorphic to the open unit ball in $\mathbb{R}^n$,
   - $V_i \cap V_{i+1} \neq \emptyset$,
   - $V_i \cap V_j = \emptyset$ if $|i - j| > 1$,
   - $\{V_i | i \in \mathbb{N}\}$ is locally finite.

   [The collection $\{V_i | i \in \mathbb{N}\}$ need not cover M.]

   (ii) Use part (i) to prove that there exists a closed embedding of the real line in every connected non-compact smooth manifold. ("Closed" means the image is closed, so you cannot simply use the fact that $\mathbb{R}$ is diffeomorphic to the interval $(0, 1)$, and embed the interval in M.)

   (iii) If M is compact, is there a closed embedding of the real line in M?

5. (i) Let $G \subset \mathbb{R}$ be a closed subset and subgroup of $(\mathbb{R}, +)$. Show that either $G = 0$, $G \cong \mathbb{Z}$, or $G = \mathbb{R}$.

   (ii) Let $\alpha_x : \mathbb{R} \to M$ be a flow line of a global flow. Show that $G := \{t \in \mathbb{R} | \alpha_x(t) = x\}$ is a closed subgroup of $(\mathbb{R}, +)$.

   (iii) Relate parts (i) and (ii) to the lemma from class which classified the behaviour of flow lines; ie. which groups G give rise to constant flows, periodic immersions, and injective immersions?

6. Show that for each vector field X on M, there is a strictly positive continuous function $\epsilon : M \to \mathbb{R}$ such that ϵX is globally integrable. [Hint: There exists a local flow for X, defined on an open subset $A \in \mathbb{R} \times M$ with $A \cap (\mathbb{R} \times \{x\}) = (a_x, b_x)$ an open interval containing 0, for each $x \in M$. Define $\epsilon(x) := \min(-a_x, b_x)/2$.]

7. Use a partition of unity to show that every submanifold of M diffeomorphic to S^1 is the orbit of some global flow on M.

8. (i) Let M be a compact manifold of dimension at least two. Find an injective immersion $\mathbb{R} \to M$ whose image is not a flow line of a flow on M.

   (ii) Using the result of question 4 (ii), show that on every non-compact connected manifolds there exists a vector field which is not globally integrable.

9. (i) For each $n \geq 0$, find a flow on S^1 with exactly n fixed points.

   (ii) Find a fixed point free flow on the odd-dimensional sphere S^{2n-1}. Which orbits are periodic? [Hint: Use the embedding $S^{2n-1} \subset \mathbb{C}^n$.]

   (iii) Find a flow on S^2 with exactly two fixed points and exactly one closed periodic orbit.

   (iv) Find a flow on $\mathbb{RP}^2$ with exactly one fixed point and with all other orbits being closed and periodic.

   [You should try to write explicit equations for these flows; don't just draw pictures.]

10. Let X be a vector field on $S^2 \subset \mathbb{R}^3$ which is never tangential to the equator. Show that each flow line can meet the equator at most once.

11. Let $g : \mathbb{R} \to \mathbb{R}_+$ be a strictly positive continuous function with $\lim_{|x| \to \infty} g(x) = 0$, and let $A := \{(t, x) \in \mathbb{R}^2 | t < g(x)\}$. Show that there is a local flow on $\mathbb{R}$ which is define on A but does not extend to a global flow.

12. Let M be a connected manifold of dimension at least two. Let $x_1, \ldots, x_k$ be distinct points on M, and let $y_1, \ldots, y_k$ also be distinct points on M (the y_i need not be distinct from the x_j). Prove that there exists a diffeomorphism $\phi : M \to M$ with $\phi(x_i) = y_i$ for all $i = 1, \ldots, k$.

13. Let M be a closed submanifold of the connected manifold N, with codimension at least two, and let p and q be points of N which don't belong to M. Show that there exists a diffeomorphism $\phi : N \to N$ with $\phi(p) = q$ and $\phi|_M = \mathrm{Id}_M$.

14. Let $E \to X$ be a smooth vector bundle. Show that every smooth section s of E is an embedding $s : X \to E$ which is isotopic to the zero-section.

15. Show that the antipodal map

$$S^n \to S^n, \qquad x \mapsto -x$$

    is isotopic to the identity if and only if n is odd.