

MAT 566 Differential Topology : Exercise Sheet Three

1. Let $\mathbb{C}\mathbb{P}^n$ be complex projective space and define

$$H(m, n) := \{(z, w) \in \mathbb{C}\mathbb{P}^m \times \mathbb{C}\mathbb{P}^n \mid \sum_{i=0}^m z_i w_i = 0\}$$

for $m \leq n$, where $z = [z_0, \dots, z_m]$ and $w = [w_0, \dots, w_n]$ are homogeneous coordinates. Show that $H(m, n)$ is a $2(m+n-1)$ -dimensional manifold. [The same construction works for real projective space. These are known as *Milnor manifolds*.]

2. We can regard the set $U(n)$ of unitary matrices (complex $n \times n$ matrices A such that the conjugate transpose \bar{A}^t equals the inverse A^{-1}) as a subset of the orthogonal group $O(2n)$. Show that $U(n)$ is a submanifold of dimension n^2 .
3. Let M , N , and L be smooth manifolds and

$$\begin{array}{ccc} M & & L \\ & f \searrow & \swarrow g \\ & N & \end{array}$$

a pair of transverse smooth maps, i.e. for all points $p \in M$ and $q \in L$ with $f(p) = g(q) = r \in N$ we have

$$T_p f(T_p M) + T_q g(T_q L) = T_r N.$$

Show that the *fibre product*

$$M \times_N L := \{(p, q) \in M \times L \mid f(p) = g(q)\}$$

of f and g is a smooth manifold.

4. Let $f : M \rightarrow N \times \mathbb{R}^n$ be a smooth map. Show that there exists an arbitrarily small vector $v \in \mathbb{R}^n$ such that the map

$$g : M \rightarrow N \times \mathbb{R}^n, \quad x \mapsto f(x) + v$$

is transverse to the submanifold $N \times \{0\} \subset N \times \mathbb{R}^n$. (In other words, for each $\epsilon > 0$ there exists v with $|v| < \epsilon$ such that... etc.)

5. (i) Let $M^n \subset \mathbb{R}^p$ be a smooth manifold embedded in Euclidean space. Show that there exists a hyperplane in \mathbb{R}^p which cuts M^n transversely.
- (ii) Let M^n be a compact manifold and $f : M^n \rightarrow \mathbb{R}^{n+1}$ a smooth map such that the image $f(M^n)$ does not contain the origin. Show that there exists a line through the origin which meets $f(M^n)$ in only a finite number of points.

6. Show that there is no surjective smooth map $\mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$.
7. (i) Let $f : M \rightarrow \mathbb{R}^p$ be a smooth map and $N \subset \mathbb{R}^p$ a smooth submanifold. Show that there exists an arbitrarily small vector $v \in \mathbb{R}^p$ such that the map $x \mapsto f(x) + v$ is transverse to N . [Hint: Find a sufficiently small regular value for the map $M \times N \rightarrow \mathbb{R}^p$ which take (x, y) to $y - f(x)$.]
- (ii) For a smooth map $f : M \rightarrow N$ define

$$\Sigma^i(f) := \{p \in M \mid \text{rk}_p f = i\}.$$

Fix i . Given a smooth map $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $\epsilon > 0$, show that there exists a linear map $\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n$ of norm less than ϵ such that $\Sigma^i(f + \alpha)$ is a smooth submanifold of \mathbb{R}^m .

(iii) Suppose $2m \leq n$. Given a smooth map $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $\epsilon > 0$, show that there exists a linear map $\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n$ of norm less than ϵ such that $f + \alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is an immersion. [Hint: Show that this is a corollary of part (ii).]

8. Let M^k be a compact manifold embedded in Euclidean space \mathbb{R}^{n+1} , with $n \geq 2k$. Show that there exists a hyperplane H in \mathbb{R}^{n+1} such that the composition

$$M^k \xrightarrow{i} \mathbb{R}^{n+1} \xrightarrow{\pi} H$$

is an immersion, where $i : M^k \hookrightarrow \mathbb{R}^{n+1}$ is inclusion and $\pi : \mathbb{R}^{n+1} \rightarrow H$ is projection. [Hint: Define the *projectivization* of the tangent bundle TM to be the $(2k - 1)$ -dimensional manifold PTM whose elements are lines in the tangent spaces of M . Consider the map $PTM \rightarrow \mathbb{R}P^n$.]

[This shows that if we can find an embedding of the compact manifold M^k in Euclidean space \mathbb{R}^N for some large N , then in fact we can immerse it in \mathbb{R}^{2k} (using induction).]

9. Let M^k be a compact manifold embedded in Euclidean space \mathbb{R}^{n+1} , with $n \geq 2k + 1$. Show that there exists a hyperplane H in \mathbb{R}^{n+1} such that the composition

$$M^k \xrightarrow{i} \mathbb{R}^{n+1} \xrightarrow{\pi} H$$

is an embedding, where $i : M^k \hookrightarrow \mathbb{R}^{n+1}$ is inclusion and $\pi : \mathbb{R}^{n+1} \rightarrow H$ is projection.

[This shows that if we can find an embedding of the compact manifold M^k in Euclidean space \mathbb{R}^N for some large N , then in fact we can embed it in \mathbb{R}^{2k+1} (using induction).]