MAT 542 Complex Analysis I : Exercise Sheet Six

1. Calculate the following integrals:

   a) $\int_0^\infty \frac{(\log x)^2}{1+x^2} dx = 0$ b) $\int_0^{\pi/2} \frac{1}{1+\sin^2\theta} d\theta$

   c) $\int_0^\pi \log \sin\theta \, d\theta$

2. Let (S, d) be a metric space. Prove that

   $$\mu(s,t) = \frac{d(s,t)}{1+d(s,t)}$$

   is also a metric on S. Prove that a set is open in (S, d) iff it is open in (S, μ). Prove that a sequence is Cauchy in (S, d) iff it is Cauchy in (S, μ).

3. Suppose that $\{f_n\}$ is a sequence in $C(G, \mathbb{C})$ which converges to f and $\{z_n\}$ is a sequence in G which converges to $z \in G$. Prove that $\lim_{n\to\infty} f_n(z_n) = f(z)$.

4. [Dini's Theorem] Suppose $\{f_n\}$ is a sequence of real-valued functions in $C(G, \mathbb{R})$ which is monotonically increasing, i.e. $f_n(z) \leq f_{n+1}(z)$ for all n and all $z \in G$. Suppose there exists $f \in C(G, \mathbb{R})$ such that $f(z) = \lim_{n\to\infty} f_n(z)$ for all $z \in G$. Prove that $f_n \to f$.

5. Suppose $\{f_n\}$ is a sequence in $C(G, \mathbb{C})$ such that the family $\{f_n\}$ is equicontinuous at each point of G. Suppose there exists $f \in C(G, \mathbb{C})$ such that $f(z) = \lim_{n\to\infty} f_n(z)$ for all $z \in G$. Prove that $f_n \to f$.

6. [Vitali's Theorem] Suppose the family $\{f_n\} \subset H(G)$ is locally bounded and there exists $f \in H(G)$ such that

   $$A = \{z \in G \,|\, \lim_{n\to\infty} f_n(z) = f(z)\}$$

   has a limit point in G. Prove that $f_n \to f$.

7. Let $f_n(z) = \tan(nz)$. Show that f_n converges uniformly to the constant function $-i$ on any compact subset of the lower half-plane $G = \{z \,|\, \text{Im} z < 0\}$.

8. Show that if $\mathcal{F} \subset H(G)$ is normal then $\mathcal{F}' := \{f' \,|\, f \in \mathcal{F}\}$ is also normal. Prove the converse or find a counter-example.

9. Prove that $\mathcal{F} \subset C(G, \mathbb{C})$ is normal iff its closure $\overline{\mathcal{F}}$ is compact.

10. Prove that $\mathcal{F} \subset C(G, \mathbb{C})$ is normal iff for all $\delta > 0$ and compact $K \subset G$ there exists finitely many functions $f_1, \ldots, f_n \in \mathcal{F}$ such that if $f \in \mathcal{F}$ then

    $$\sup_{z\in K} |f(z) - f_k(z)| < \delta$$

    for at least one $k \in \{1, \ldots, n\}$. In other words, $\mathcal{F}$ is covered by balls of radius δ centred at the f_i.

11. Let (X_n, d_n) be metric spaces and $X = \prod_{n=1}^{\infty} X_n$ with the metric

$$d(\xi, \eta) := \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}$$

where $\xi = \{x_n\}$ and $\eta = \{y_n\}$.

a) Prove that (X, d) is a metric space.

b) Let $\xi^k = \{x_n^k\} \in X$. Prove that $\lim_{k\to\infty} \xi^k = \xi = \{x_n\}$ iff $\lim_{k\to\infty} x_n^k = x_n$ for all n.

c) Show that (X, d) is compact if each (X_n, d_n) is compact.

12. Prove that $\mathcal{F} \subset H(G)$ is locally bounded if it is uniformly bounded on each compact $K \subset G$.

13. Let $\mathcal{F}$ be the family of functions on G whose real part is positive at all points of G. Prove that $\mathcal{F}$ is normal. Is $\mathcal{F}$ locally bounded? Prove or find a counter-example.

14. Prove that the family of functions $\mathcal{F} = \{z^n \,|\, n \in \mathbb{Z}_{n\geq 0}\}$ is normal in both $C(B(0;1), \mathbb{C})$ and $C(\mathbb{C} - \overline{B(0;1)}, \mathbb{C})$. However, if G contains a point of the unit circle then $\mathcal{F}$ is *not* normal in $C(G, \mathbb{C})$. Why not?

15. Let f be an entire function and let $\mathcal{F} = \{f(kz) \,|\, k \in \mathbb{C} \text{ constant}\}$. Show that $\mathcal{F}$ is normal in $C(B(0;R) - \overline{B(0;r)}, \mathbb{C})$ iff f is a polynomial.

16. Show that there does *not* exist a one-to-one analytic function which maps the punctured disc $B(0;1) - \{0\}$ onto an annulus $B(0;R) - \overline{B(0;r)}$ with $r > 0$. [Hint: Show that 0 would be a removable singularity of such a map.]

17. Show that the annuli

$$B(0;R_1) - \overline{B(0;r_1)} \qquad \text{and} \qquad B(0;R_2) - \overline{B(0;r_2)}$$

are conformally equivalent if $R_1/r_1 = R_2/r_2$. Is this condition also necessary? Prove or find a counter-example.

18. Find an analytic function which maps the upper half-disc

$$B(0;1) \cap \{z \,|\, \mathrm{Im} z > 0\}$$

bijectively to $B(0;1)$.

19. Let f be a one-to-one analytic map from the upper half-plane $H = \{z \,|\, \mathrm{Im} z > 0\}$ *into* H. If $ai \in i\mathbb{R}_{>0}$ is fixed by f, i.e. $f(ai) = ai$, prove that $|f'(ai)| \leq 1$.

20. Let G be a bounded region whose boundary consists of two non-intersecting circles. Prove that there is a one-to-one conformal mapping from G onto an annulus.