MAT 542 Complex Analysis I : Exercise Sheet Three

1. Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth curve and f a function continuous on $\gamma([a, b]) \subset \mathbb{C}$. Prove that

$$\left| \int_\gamma f dz \right| \leq \int_\gamma |f||dz| \leq V(\gamma) \sup_{t\in[a,b]} |f(\gamma(t))|$$

where $V(\gamma) := \int_\gamma |dz|$ is the length of γ.

2. a) Let $\gamma(t) = \exp((-1 + i)t^{-1})$ for $t \in (0, 1]$ and $\gamma(0) = 0$. Sketch $\gamma([0, 1]) \subset \mathbb{C}$ and find $V(\gamma)$.

   b) Let $\gamma(t) = t + it \sin(t^{-1})$ for $t \in (0, 1]$ and $\gamma(0) = 0$. Sketch $\gamma([0, 1]) \subset \mathbb{C}$ and show that $V(\gamma)$ does not exist (i.e. is infinite).

3. a) Compute $\int_\gamma x dz$ where γ is the straight line from 0 to $1 + i$.

   b) Compute $\int_{|z|=2} \frac{dz}{z^2-1}$.

   c) Compute $\int_{|z|=1} |z - 1||dz|$.

4. Compute $\int_{|z|=r} x dz$ in two ways: first by using a parameter, and second by observing that

$$x = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}(z + \frac{r^2}{z})$$

   on the circle.

5. Let γ be the straight line from 1 to i, and let σ consist of the straight line from 1 to $1+i$ *and* the straight line from $1 + i$ to i. Calculate and compare $\int_\gamma |z|^2 dz$ and $\int_\sigma |z|^2 dz$.

6. a) Let $I(r) := \int_\gamma \frac{e^{iz}}{z} dz$ where $\gamma : [0, \pi] \to \mathbb{C}$ is given by $\gamma(t) = re^{it}$. Show that $\lim_{r\to\infty} I(r) = 0$.

   b) Now suppose that $\gamma : [0, 2\pi] \to \mathbb{C}$ is given by the same equation. Find $\lim_{r\to\infty} I(r)$.

7. Find $\int_\gamma z^{-1/2} dz$ where

   a) γ is the upper half of the unit circle from 1 to -1,

   b) γ is the lower half of the unit circle from 1 to -1.

8. Find $\int_\gamma (z^2 - 1)^{-1} dz$ where

   a) $\gamma(t) = 1 + e^{it}$ for $t \in [0, 2\pi]$,

   b) $\gamma(t) = 2e^{it}$ for $t \in [-\pi, \pi]$.

9. If f is analytic, show that

$$\int_\gamma \overline{f(z)} f'(z) dz$$

   is purely imaginary for all closed paths γ.

10. Find a closed path γ such that for any integer k there exists a point $a \in \mathbb{C}$, not on γ, such that $n(\gamma; a) = k$. Can $V(\gamma)$ be finite?

11. Let $p(z)$ be a polynomial of degree n which has no zeros outside of the circle $\gamma(t) = Re^{it}$, $t \in [0, 2\pi]$. Prove that
$$\int_\gamma \frac{p'(z)}{p(z)} dz = 2\pi i n.$$

12. Let $w = re^{i\theta}$ be non-zero, and let γ be a path from 1 to w which does not pass through zero. Show that there is an integer k such that
$$\int_\gamma \frac{dz}{z} = \log r + i\theta + 2\pi i k.$$

13. Let f be analytic on the unit ball $B(0;1)$ and suppose that $|f(z)| \leq 1$ for $|z| < 1$. Prove that $|f'(0)| \leq 1$.

14. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function which is analytic on $\mathbb{C} - [-1, 1]$. Prove that in fact f must be analytic on all of $\mathbb{C}$.

15. Let f be a function analytic in a neighbourhood of zero. Show that f *cannot* satisfy
$$|f^{(n)}(0)| > n! n^n$$
for all n. Formulate an "optimal" inequality that *can* be satisfied for all n.