MAT 542 Complex Analysis I : Exercise Sheet Two

In the following problems, $B(a;r)$ denotes the open disc with centre a and radius r.

1. Find the images of the following regions under the exponential map:

   a) $\{z \,|\, \mathrm{Re} z < 0,\ |\mathrm{Im} z| < \pi\}$, b) $\{z \,|\, \mathrm{Re} z < 0,\ |\mathrm{Im} z| < \frac{\pi}{2}\}$, c) $\{z \,|\, |\mathrm{Im} z| < \frac{\pi}{2}\}$.

2. A Möbius transformation that fixes three points must be the identity. How many points are fixed by translations, dilations, rotations, and the inversion, respectively?

3. a) Suppose that $Tz = \frac{az+b}{cz+d}$ fixes 0 and ∞. Show that T is a dilation (or the identity).

   b) Suppose that $Tz = \frac{az+b}{cz+d}$ fixes only ∞. Show that T is a (non-trivial) translation.

   c) Suppose that $Tz = \frac{az+b}{cz+d}$ interchanges 0 and ∞. Show that, up to a constant, T is the inversion.

4. Let S and T be non-trivial Möbius transformations. Prove that they commute, $ST = TS$, if they have the same fixed points.

5. If $Tz = \frac{az+b}{cz+d}$, find z_2, z_3, and z_4 (in terms of a, b, c, and d) such that Tz equals the cross ratio (z, z_2, z_3, z_4).

6. Let z_1, z_2, z_3, and z_4 be four general points in the extended complex plane, and let $\lambda = (z_1, z_2, z_3, z_4)$ be their cross ratio. How many *different* cross ratios do we get by permuting the four points? Express the other cross ratios in terms of λ.

7. Give four distinct points in the extended complex plane, show that there exists a Möbius transformation which takes them to 1, -1, w, and $-w$, respectively. Is the solution unique? [Hint: Write w in terms of the cross ratio of the points and use the result of the previous problem.]

8. Let $Tz = \frac{az+b}{cz+d}$ be a Möbius transformation. Find conditions on a, b, c, and d so that T preserves:

   a) the extended real line $\mathbb{R} \cup \infty$,

   b) the unit circle $\{z \,|\, |z| = 1\}$,

   c) the unit disc $B(0;1)$,

   d) the upper half plane $\{z \,|\, \mathrm{Im} z > 0\}$.

9. Can $Tz = \frac{1}{2}(z + z^{-1})$ be written as the composition of Möbius transformations?

10. Find a function which conformally maps

$$B(0;2) - \overline{B(1;1)} = \{z \,|\, |z| < 2, |z-1| > 1\}$$

    onto the unit disc $B(0;1)$. Can such a function be a Möbius transformation?

11. Find a function which conformally maps the unit disc $B(0;1)$ onto the first quadrant $\{z \,|\, \mathrm{Re} z > 0, \mathrm{Im} z > 0\}$. Can such a function be a Möbius transformation?

12. Find a function which conformally maps the half disc

$$B(0;1) \cap \{z \mid \mathrm{Re} z > 0\} = \{z \mid |z| < 1, \mathrm{Re} z > 0\}$$

onto the first quadrant $\{z \mid \mathrm{Re} z > 0, \mathrm{Im} z > 0\}$. Can such a function be a Möbius transformation?

13. Let $Tz = \frac{az+b}{cz+d}$ and $Sz = \frac{\alpha z+\beta}{\gamma z+\delta}$. Show that S and T represent the same map if and only if there is a non-zero $\lambda \in \mathbb{C}$ such that $a = \lambda\alpha$, $b = \lambda\beta$, $c = \lambda\gamma$, and $d = \lambda\delta$.

14. For $|z| < 1$ let

$$f(z) = \exp\left(-i\log\left[i\left(\frac{1+z}{1-z}\right)\right]^{1/2}\right).$$

Show that f conformally maps the unit disc $B(0;1)$ onto an annulus. Why does this *not* contradict the fact that $B(0;1)$ is simply-connected while an annulus is not?