# MAT 542 Complex Analysis I : Exercise Sheet One

1. a) For $z$ and $z' \in \mathbb{C}$, show that

$$d(z, z') = \frac{2|z - z'|}{[(1 + |z|^2)(1 + |z'|^2)]^{1/2}}.$$

   b) Show that $(\mathbb{C}_\infty, d)$ is a metric space.

2. a) Show that $|z - z_n| \to 0$ iff $d(z, z_n) \to 0$ for $z$ and $z_n \in \mathbb{C}$.

   b) Show that if $|z_n| \to \infty$ then $\{z_n\}$ is a Cauchy sequence in $\mathbb{C}_\infty$. Does this mean $\{z_n\}$ converges in $\mathbb{C}_\infty$?

3. Suppose $f : X \to \mathbb{C}$ is uniformly continuous and $\{x_n\}$ is a Cauchy sequence in $X$. Show that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{C}$. Would this be true if $f$ were only continuous? (Prove or give a counterexample.)

4. Let $D$ be a dense subset of $X$ and let $f : D \to \mathbb{C}$ be uniformly continuous. Prove that $f$ can be extended to a uniformly continuous function from $X$ to $\mathbb{C}$. Is the extension unique? Is this still possible if we only assume $f$ is continuous? (Prove or give a counterexample.)

5. Let $\{f_n\}$ be a sequence of uniformly continuous functions from $X$ to $\mathbb{C}$ which uniformly converge to $f = u - \lim f_n$. Prove that $f$ is uniformly continuous.

6. Let $f : G_1 \to G_2 \subset \mathbb{C}$ and $g : G_2 \to \mathbb{C}$ be nonconstant complex functions.

   a) Suppose that $f$ and $g \circ f$ are analytic. Must $g$ be analytic too? (Prove or give a counterexample.)

   b) Suppose that $g$ and $g \circ f$ are analytic. Must $f$ be analytic too? (Prove or give a counterexample.)

7. a) Let $f : G \to \mathbb{C}$ be an analytic function which takes values only in $\mathbb{R}$. Prove that $f$ is constant.

   b) Let $f : G \to \mathbb{C}$ be an analytic function for which $|f|$ is constant. Prove that $f$ is constant.

8. a) Let $f_1$ and $f_2 : G \to \mathbb{C}$ be two analytic functions with the same real part. Prove that $f_1$ and $f_2$ differ at most by a constant.

   b) Suppose that the real part of $f(z) = f(x + iy)$ is $x^2 - y^2$. Use the Cauchy-Riemann equations to find the harmonic function conjugate to $x^2 - y^2$, i.e. the imaginary part of $f$ (up to a constant).

   c) Find the most general harmonic polynomial of the form $ax^3 + bx^2y + cxy^2 + dy^3$. Determine the conjugate harmonic function.

9. a) Let $f(z)$ be an analytic function. Show that $\overline{f(\bar{z})}$ is also analytic.

   b) Let $u(z) = u(x+iy) = u(x, y)$ be a harmonic function. Show that $u(\bar{z})$ is also harmonic.

10. a) For which values of $z$ does $\sum \left(\frac{z}{1+z}\right)^n$ converge?

    b) For which values of $z$ does $\sum \frac{z^n}{1+z^{2n}}$ converge?

11. Let $\sum a_n z^n$ be a series with radius of convergence $R$, where

$$\frac{1}{R} = \text{limsup}|a_n|^{1/n},$$

and suppose that $\lim |a_n/a_{n+1}|$ exists. Prove that

$$R = \lim |a_n/a_{n+1}|.$$

12. Find the radii of convergence of the following series:

    a) $\sum n^p z^n$, $p \in \mathbb{Z}$, b) $\sum a^{n^2} z^n$, $a \in \mathbb{C}$, c) $\sum n! z^n$, d) $\sum z^{n!}$.

13. Show that the radius of convergence of

$$\sum \frac{(-1)^n}{n} z^{n(n+1)}$$

is one. For which of the values $z = 1, -1, i, -i$ does the series converge? For which values on the unit circle does the series converge/diverge?

14. Find the general form of a rational function which has absolute value one on the unit circle $|z| = 1$. What can you say about the zeros and poles?

15. Find the real and imaginary parts of

    a) $\cos z$, b) $\sin z$, c) $\exp(e^z)$, d) $z^z$.

16. Let $z$ and $z_n \in \mathbb{C} - \{z | z \leq 0\}$, and write $z = re^{i\theta}$ and $z_n = r_n e^{i\theta_n}$ where $\theta$ and $\theta_n$ are strictly between $-\pi$ and $\pi$. If $z_n \to z$, prove that $r_n \to r$ and $\theta_n \to \theta$.

17. Let $z \in \mathbb{C} - \{z | z \leq 0\}$. If $z^{1/2}$ is defined using the principal branch of the logarithm, prove that the real part of $z^{1/2}$ is positive.

18. Let $r > 0$. Describe the image of $B(0; r) - \{0\}$ under the map $z \mapsto \exp(z^{-1})$.

19. Prove that there is no branch of the logarithm defined on $\mathbb{C} - \{0\}$.