MAT 260 Problem Solving in Mathematics
Week 9 : Number Theory

1. Find all solutions in positive integers x, y, z, of

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 1.$$

2. Let $\{a_1, a_2, \ldots, a_n\}$ be a permutation of the integers 1 to n. If n is odd, prove that

$$(a_1 - 1)(a_2 - 2) \cdots (a_n - n)$$

must be even.

3. Suppose that both p and $2^p - 1$ are prime numbers, and let $n = 2^{p-1}(2^p - 1)$. Show that the sum of the divisors of n is equal to $2n$. (We say that n is *perfect* number; for example, 6 and 28 are perfect. It is not known whether there exist perfect numbers which are odd.)

4. (i) If n is a positive integer such that $2n + 1$ is a perfect square, show that $n + 1$ is the sum of two successive perfect squares. For example, if $n = 12$ then $2n+1 = 5^2$, and $n+1 = 2^2+3^2$.
 (ii) If $3n + 1$ is a perfect square, show that $n + 1$ is the sum of three perfect squares. For example, if $n = 16$ then $3n + 1 = 7^2$, and $n + 1 = 2^2 + 2^2 + 3^2$.

5. Suppose we have an arithmetic sequence of positive integers, with common difference d. Show that the product of four *consecutive* terms plus d^4 is a perfect square. For example, if $d = 2$ then $5.7.9.11 + 2^4 = 59^2$.

6. (i) Suppose that x and y are odd integers, and z is an even integer. Prove that they cannot be a solution of

$$x^2 + y^2 + z^2 = 2^n xyz$$

where n is a positive integer. [Hint: Consider the remainder of each side when we divide by 4.]
 (ii) Use part (i) to prove that there do not exist positive integers x, y, and z such that

$$x^2 + y^2 + z^2 = 2xyz.$$

7. (i) For positive integers $m > n$, let $x = m^2 - n^2$, $y = 2mn$, and $z = m^2 + n^2$. Show that

$$x^2 + y^2 = z^2.$$

 (ii)* Prove that all solutions of

$$x^2 + y^2 = z^2$$

are given as in part (i).

8. Prove that there is no solution in non-negative integers a, b, c, of

$$(2^a - 1)(2^b - 1) = 2^{2^c} + 1.$$

9.* Let a be a positive integer. Prove that $x^2 - y^2 = a^3$ always has at least one solution in positive integers x and y. [Hint: Consider the different cases, when the numbers are odd/even.]