## MAT 260 Problem Solving in Mathematics
## Week 8 : Calculus

1. If $x$ satisfies $x^4 + 36 \leq 13x^2$, find the maximum value of $f(x) = x^3 - 3x$.

2. A cross-country runner runs six miles in half an hour. Prove that somewhere along the course he/she ran one mile in exactly five minutes. [Hint: Intermediate value theorem.]

3. Let $f$ be a continuous function from the interval $[0, 1]$ to the real numbers, with $f(0) = f(1)$. Prove that for all positive integers $n$, there exists a real number $x \in [0, 1 - \frac{1}{n}]$ such that $f(x) = f(x + \frac{1}{n})$. [Hint: Intermediate value theorem.]

4. A rock climber sets off at 7am Saturday morning and arrives at the top of the mountain at 5pm. He camps overnight, then starts his descent at 7am Sunday morning, arriving at the bottom at 5pm. Prove that there is some time such that he was at exactly the same altitude at that time on both days. [Hint: Intermediate value theorem.]

5. Define a function by $f(x) = a_1 \sin(x) + a_2 \sin(2x) + \ldots + a_n \sin(nx)$ where $a_1$, $a_2$, $\ldots$, and $a_n$ are real numbers, and $n$ is a positive integer. If $|f(x)| \leq |\sin(x)|$ for all real $x$, prove that

$$|a_1 + 2a_2 + 3a_3 + \ldots + na_n| \leq 1.$$

[Hint: Consider the derivative of $f$ at $x = 0$. Using the definition of the derivative, can you deduce a bound on $f'(0)$?]

6. Let $a$, $b$, and $c$ be real numbers. Show that the equation

$$4ax^3 + 3bx^2 + 2cx = a + b + c$$

always has a root between 0 and 1. [Hint: Rolle's theorem.]

7. How many real solutions of the equation

$$2^x = 1 + x^2$$

are there?

8*. In the graph below, the lower function is $\frac{x^2}{2}$, the middle function is $x^2$, and the upper function is unknown and denoted $f$. Find $f$ if the area of region $A$ is the same as the area of region $B$ for all points $P$.