MAT 260 Problem Solving in Mathematics
Week 7 : Series

1. (i) Observe that $(k+1)^2 - k^2 = 2k+1$. Summing both sides gives

$$(n+1)^2 - 1^2 = ((n+1)^2 - n^2) + (n^2 - (n-1)^2) + \ldots + (3^2 - 2^2) + (2^2 - 1^2) = 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

and therefore

$$\sum_{k=1}^{n} k = \frac{(n+1)^2 - 1 - n}{2} = \frac{n(n+1)}{2}.$$

Sum $(k+1)^3 - k^3 = 3k^2 + 3k + 1$ and hence show that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

(ii) By summing $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$, find a formula for $\sum_{k=1}^{n} k^3$.

2. Show that

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots + \frac{1}{n(n+1)}$$

is a telescoping series, and hence prove that the sum is never greater than one, no matter how large n is.

3. By differentiating or integrating the series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

find

(*i*) $$1\times 3 + 2\times 3^2 + 3\times 3^3 + \ldots + n\times 3^n,$$

(*ii*) $$1\times 3^n + 2\times 3^{n-1} + 3\times 3^{n-2} + \ldots n\times 3^1,$$

(*iii*) $$\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{4}\left(\frac{1}{2}\right)^4 + \ldots$$

4. (i) Find

$$\frac{1}{0!} + \frac{2}{1!} + \frac{3}{2!} + \frac{4}{3!} + \ldots$$

[Hint: Begin with the power series for e^x, multiply by x, then differentiate.]

(ii) Find

$$\frac{1^2}{0!} + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \ldots$$

[Hint: Begin with the power series for e^x, multiply by x, then differentiate. Now multiply by x and differentiate again.]

5. The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ... is defined recursively by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Use telescoping of series to prove that

$$(i) \qquad F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1,$$

$$(ii) \qquad F_1 + F_3 + F_5 + \ldots + F_{2n-1} = F_{2n},$$

$$(iii) \qquad F_1^2 + F_2^2 + F_3^2 + \ldots + F_n^2 = F_n F_{n+1},$$

$$(iv) \qquad \frac{1}{F_1 F_3} + \frac{1}{F_2 F_4} + \frac{1}{F_3 F_5} + \frac{1}{F_4 F_6} + \ldots = 1$$

6*. Find the limit of the sum

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n-1}$$

as $n \to \infty$. [Hint: Find an upper and lower bound on the sum by integrating x^{-1} between certain values.]