MAT 260 Problem Solving in Mathematics
Week 6 : The Pigeonhole Principle

1. (i) Suppose we are given 20 different numbers from the arithmetic progression

$$1, \quad 4, \quad 7, \quad 10, \quad \ldots \quad 100.$$

Out of the 20 numbers that we are given, prove that two of them add up to 104.

(ii) Suppose we are given 10 different numbers from $\{1, 2, 3, \ldots 99\}$. Prove that there exists two disjoint subsets with the same sum. For example, if we are given the ten numbers $\{1, 2, 7, 11, 23, 31, 35, 48, 83, 91\}$ then

$$2 + 11 + 23 = 1 + 35.$$

[Hint: In how many ways can we choose a subset of our ten numbers? How many different sums are there?]

2. (i) Given five points in a square of side length 1, show that two of the points must be no further than $\sqrt{2}/2$ apart.

(ii) Given nine points in a square of side length 2, show that three of the points must form a triangle whose area is not more that $1/2$.

3. (i) There are n people at party. Prove that two of them know exactly the same number of people (among those present).

(ii) There are six people at a party. For each pair of people, either they met before or they didn't. Prove that either there are three people who met each other before, or three people who never met each other before (or both).

(iii)* Seventeen cities are served by Delta Airlines, Northwest Airlines, and United Airlines, such that between any two cities exactly one of the airlines flies. Prove that there are three cities all connected by the same airline (i.e. either all connected by Delta, or all connected by Northwest, or all connected by United).

4. Suppose I have a convex pentagon (i.e. all internals angles are less then π radians) in the plane, whose vertices lie at lattice points (i.e. the coordinates of the vertices are integers). Prove that the pentagon must contain at least one lattice point.

5^*. Suppose we are given $n + 1$ different numbers from $\{1, 2, 3, \ldots, 2n\}$. Prove that one of the numbers divides another one.

6^*. We are given a 3×7 chessboard, all of whose squares are either white or black. Prove that there is a rectangle whose corners are either all white or all black (as shown in the picture).