MAT 260 Problem Solving in Mathematics
Week 5 : Inequalities

1. (i) Use the inequality

$$\frac{a+b}{2} \geq (ab)^{1/2}$$

to prove the inequality

$$\frac{a+b+c+d}{4} \geq (abcd)^{1/4},$$

where a, b, c, and d are positive real numbers.

(ii) Prove the inequality

$$\frac{a+b+c}{3} \geq (abc)^{1/3}.$$

[Hint: Put $d = (a+b+c)/3$.]

(iii) Prove the inequality

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n}.$$

2. (i) A farmer wants to make a rectangular pen using 100 metres of fencing. What dimensions will give the greatest area, and what is that area?

(ii)* Now suppose the pen is triangular. What kind of triangle will give the greatest area, and what is that area?

(iii) Now suppose the pen can be any shape. What shape will give the greatest area, and what is that area?

3. A right-angled triangle has hypotenuse of length c, while the other sides have lengths a and b. Show that

$$a + b \leq (\sqrt{2})c.$$

4. Let a and b be positive numbers with $a + b = 1$. Prove that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}.$$

When do we have equality?

5. Let a, b, and c be positive numbers. Prove that

$$(a+b)(b+c)(c+a) \geq 8abc$$

with equality only when $a = b = c$.

6. Let a_1, a_2, …, a_n be positive numbers. Prove that

$$(a_1 + a_2 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \geq n^2.$$

When does equality hold?

7. Prove that for any integer $n > 1$

$$n! < \left(\frac{n+1}{2}\right)^n.$$

8. I have a piece of cardboard measuring 30cm on each side. I want to make a box as shown, by cutting a square of size x cm from each corner and bending up the edges. What choice of x gives the largest volume? [Hint: If you can't solve this using inequalities, use calculus instead.]