MAT 260 Problem Solving in Mathematics
Week 4 : Generating Functions

1. (i) A generating function for partitions of n is

$$\prod_{k\geq 1} \frac{1}{1-x^k}.$$

Find a generating function for the number of partitions of n into even numbers.

(ii) Find a generating function for the number of partitions of n into odd numbers.

(iii) Find a generating function for the number of partitions of n into distinct numbers.

(iv)* Show that the generating functions from (ii) and (iii) are the same. In other words, the number of partitions of n into odd numbers is the same as the number of partitions of n into distinct numbers.

(v) Find a generating function for the number of partitions of n into numbers less than or equal to m.

(vi)* Find a generating functions for the number of partitions of n into at most m numbers.

2. (i) Find a generating function, like in question 1 part (iii), that gives the number of partitions of n into distinct powers of two. Show that your function is equal to

$$1 + x + x^2 + x^3 + x^4 + \ldots$$

Why is this the case? [Hint: Consider writing an integer in binary, for example 23 is 10111 in binary. In how many ways can I write 23 in binary?]

(ii) Find a generating function, like in question 1 part (iii), that gives the number of partitions of n into powers of three, with each power occurring at most twice. Show that your function is again equal to

$$1 + x + x^2 + x^3 + x^4 + \ldots$$

Why is this the case?

(iii) Find a generating function, like in question 1 part (iii), that gives the number of partitions of n into powers of four, with each power occurring at most three times. Once again your function should equal

$$1 + x + x^2 + x^3 + x^4 + \ldots$$

3. Find a pair of dice such that when you roll them and add together their values you get the same numbers and probabilities as with a normal pair of dice, but which are not normal dice. The numbers on the dice should all be positive integers, but they need not be ≤ 6 and they can be repeated. [Hint: If the first dice has the numbers a_1, a_2, ..., a_6 on its faces, and the second dice has the numbers b_1, b_2, ..., b_6, then we should have

$$(x^{a_1} + x^{a_2} + \ldots + x^{a_6})(x^{b_1} + x^{b_2} + \ldots + x^{b_6}) = (x + x^2 + \ldots + x^6)(x + x^2 + \ldots + x^6).$$

Why is this the case?]