MAT 260 Problem Solving in Mathematics
Week 3 : The Binomial Theorem

Use the Binomial Theorem to evaluate the following sums (where n is a positive integer):

1. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$

2. $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n}$

3. $\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \ldots + \frac{1}{n+1}\binom{n}{n}$

4. $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots + n\binom{n}{n}$

5. $\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \ldots + (-1)^{n-1} n\binom{n}{n}$

6*. $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2$

7*. $\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \ldots + (-1)^n \binom{n}{n}^2$

8. $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \ldots$

9. $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \binom{n}{7} + \ldots$

10. $\binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \binom{n}{12} + \ldots$

11. $\binom{n}{1} + \binom{n}{5} + \binom{n}{9} + \binom{n}{13} + \ldots$

12*. $\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \binom{n}{9} + \ldots$

13. Consider Pascal's triangle

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

where each entry is the sum of the two closest entries from the row above. Prove that the k^{th} entry of the n^{th} row is $\binom{n}{k}$ (we will call the first row the 0^{th} row). [Hint: Using induction this reduces to an identity involving three binomial coefficients, which is easily proved by writing them in terms of factorials.]

14*. Consider the following triangle of numbers

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & 1 & 1 & & & \\ & & 1 & 2 & 3 & 2 & 1 & & \\ & 1 & 3 & 6 & 7 & 6 & 3 & 1 & \\ 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \end{array}$$

where each entry is the sum of the three closest entries from the row above. The n^{th} row contains $2n+1$ numbers, which we shall call B_0^n, B_1^n, B_2^n, ..., and B_{2n}^n, where again we call first row the 0^{th} row. Prove that

- $B_0^n + B_1^n + B_2^n + \ldots + B_{2n}^n = 3^n$,
- $B_0^n - B_1^n + B_2^n - \ldots + B_{2n}^n = 1$,
- $(B_0^n)^2 + (B_1^n)^2 + (B_2^n)^2 + \ldots + (B_{2n}^n)^2 = B_{2n}^{2n}$.

[Hint: These sums look quite similar to those in problems 1, 2, and 6, so can you find an analogue of the Binomial Theorem? In other words, try guessing a formula for

$$B_0^n a^{2n} + B_1^n a^{2n-1} b + B_2^n a^{2n-2} b^2 + \ldots + B_{2n}^n b^{2n},$$

then prove it using, for instance, induction.]