

MAT 260 Problem Solving in Mathematics
Week 11 : Algebra

1. (i) Suppose that $a + b + c = 0$. Show that a , b , and c are the zeros of a cubic polynomial which looks like

$$p(x) = x^3 + Ax - B$$

i.e. with no x^2 term. Write A and B in terms of a , b , and c .

(ii) Let $T_n = a^n + b^n + c^n$. Then $T_0 = 1 + 1 + 1 = 3$ and $T_1 = a + b + c = 0$. Write T_2 in terms of A .

(iii) Now find a recursion relation for T_n . [Hint: Since a is a zero of $p(x)$, we have $a^3 = -Aa + B$. Multiplying by a^n gives $a^{n+3} = -Aa^{n+1} + Ba^n$. The same goes for b and c , and summing gives a formula for T_{n+3} in terms of T_{n+1} and T_n .]

(iv) Use the recursion relation to find T_3, T_4, T_5, T_6 , and T_7 in terms of A and B .

(v) Notice the relations

$$\left(\frac{a^2 + b^2 + c^2}{2}\right) \left(\frac{a^3 + b^3 + c^3}{3}\right) = \left(\frac{T_2}{2}\right) \left(\frac{T_3}{3}\right) = \frac{T_5}{5} = \frac{a^5 + b^5 + c^5}{5}$$

and

$$\left(\frac{a^2 + b^2 + c^2}{2}\right) \left(\frac{a^5 + b^5 + c^5}{5}\right) = \left(\frac{T_2}{2}\right) \left(\frac{T_5}{5}\right) = \frac{T_7}{7} = \frac{a^7 + b^7 + c^7}{7}.$$

Can you find any more formulae like these?

2. Prove that in the product

$$(1 - x + x^2 - x^3 + \dots - x^9 + x^{10})(1 + x + x^2 + x^3 + \dots + x^9 + x^{10})$$

after multiplying and collecting terms, there does not appear a term in x of odd degree.

3. Find the sum of the coefficients of

$$(1 - 2x + 2x^2)^{10}(1 + 2x - 2x^2)^{10}$$

after expanding and collecting terms.

4. (i) Find the remainder (a number) when we divide the polynomial $x + x^4 + x^9 + x^{16} + x^{25} + x^{36}$ by $x - 1$.

(ii) Find the remainder (a linear function of the form $ax + b$) when we divide $x + x^4 + x^9 + x^{16} + x^{25} + x^{36}$ by $x^2 - 1$.

5. The polynomial $p(x)$ gives a remainder of 2 when divided by $x - 1$, and a remainder of 1 when divided by $x - 2$. What is the remainder when we divide $p(x)$ by $(x - 1)(x - 2)$? Your answer should be of the form $ax + b$.

6. (i) Find a polynomial with integer coefficients which has $\sqrt{2} + \sqrt{3}$ as a root.
(ii) Find a polynomial with integer coefficients which has $\sqrt{2} + 3^{\frac{1}{3}}$ as a root.
(iii) Find a polynomial with integer coefficients which has $2^{\frac{1}{3}} + 3^{\frac{1}{3}}$ as a root.

7. For which values of a do the quadratics

$$x^2 + ax + 1 \quad \text{and} \quad x^2 + x + a$$

have a common zero?

8. Suppose that

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

is a polynomial with integer coefficients such that $p(0)$ and $p(1)$ are both odd. Prove that $p(x) \neq 0$ for all integers x . [Hint: Consider the cases x is odd/even separately.]

9. Find a polynomial $p(x)$ with rational coefficients such that $p(x^2)$ is an integer whenever x is an integer, but there exists an integer x such that $p(x)$ is not an integer.