The Schottky Problem

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What is the Schottky problem?

Schottky problem is the following question (over $\mathbb{C}$):

Which principally polarized abelian varieties
are Jacobians of curves?

$\mathcal{M}_g$	moduli space of curves C of genus g
$\mathcal{A}_g$	moduli space of g-dimensional abelian varieties (A, Θ) (complex principally polarized)
$Jac : \mathcal{M}_g \hookrightarrow \mathcal{A}_g$	Torelli map
$\mathcal{J}_g := Jac(\mathcal{M}_g)$	locus of Jacobians

(Recall that A is a projective variety with a group structure; Θ is an ample divisor on A with $h^0(A, \Theta) = 1$; $Jac(C) = \mathrm{Pic}^{g-1}(C) \simeq \mathrm{Pic}^0(C)$)

Schottky problem.

Describe/characterize $\mathcal{J}_g \subset \mathcal{A}_g$.

Why might we care about the Schottky problem?

- Relates two important moduli spaces. Lots of beautiful geometry arises in this study. A "good" answer could help relate the geometry of $\mathcal{M}_g$ and $\mathcal{A}_g$.
- Could have applications to problems about curves easily stated in terms of the Jacobian:

Coleman's conjecture

For g sufficiently large there are finitely many curves of genus g such that their Jacobians have complex multiplication.

Stronger conjecture (+ Andre-Oort $\Longrightarrow$ Coleman).

There do not exist any complex geodesics for the natural metric on $\mathcal{A}_g$ that are contained in $\overline{\mathcal{J}_g}$ (and intersect $\mathcal{J}_g$).
[Work on this by `Möller-Viehweg-Zuo; Hain, Toledo...`]

- (Super)string scattering amplitudes [`D'Hoker-Phong`], ...

Dimension counts

g	$\dim \mathcal{M}_g$		$\dim \mathcal{A}_g$	
1	1	$=$	1	
2	3	$=$	3	$\mathcal{M}_g = \mathcal{A}_g^{indecomposable}$
3	6	$=$	6	
4	9	$+1 =$	10	Schottky's original equation
5	12	$+3 =$	15	Partial results
g	$3g-3$	$+\frac{(g-3)(g-2)}{2} =$	$\frac{g(g+1)}{2}$	"weak" solutions (up to extra components)

Classical (Riemann-Schottky) approach

Embed $\mathcal{A}_g$ into $\mathbb{P}^N$ and write equations for the image of $\mathcal{J}_g$.

$\mathcal{H}_g :=$ Siegel upper half-space of dimension g
$= \{\tau \in Mat_{g\times g}(\mathbb{C}) \mid \tau^t = \tau,\ \mathrm{Im}\tau > 0\}$.
Given $\tau \in \mathcal{H}_g$, have $A_\tau := \mathbb{C}^g/(\tau\mathbb{Z}^g + \mathbb{Z}^g) \in \mathcal{A}_g$.

For $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g,\mathbb{Z})$ let $\gamma \circ \tau := (C\tau + D)^{-1}(A\tau + B)$.
Claim: $\mathcal{A}_g = \mathcal{H}_g/\,\mathrm{Sp}(2g,\mathbb{Z})$.

Definition

A modular form of weight k with respect to $\Gamma \subset \mathrm{Sp}(2g,\mathbb{Z})$ is a function $F : \mathcal{H}_g \to \mathbb{C}$ such that

$$F(\gamma\circ\tau) = \det(C\tau + D)^k F(\tau) \qquad \forall\gamma \in \Gamma, \forall\tau \in \mathcal{H}_g$$

Definition

For $\varepsilon, \delta \in \frac{1}{n}\mathbb{Z}^g/\mathbb{Z}^g$ (or $m = \tau\varepsilon + \delta \in A_\tau[n]$) the theta function with characteristic ε, δ or m is

$$\theta_m(\tau, z) := \sum_{N\in\mathbb{Z}^g} \exp\left[\pi i(N+\varepsilon, \tau(N+\varepsilon)) + 2\pi i(N+\varepsilon, z+\delta)\right]$$

- As a function of z, $\theta_m(\tau, z)$ is a section of $t_m\Theta$ (t_m =translate by m) on A_τ, so $\theta_m(\tau, z)^n$ is a section of $n\Theta$.
- For $n = 2$, $\theta_m(\tau, z)$ is even/odd in z depending on whether $4\varepsilon\cdot\delta$ is even/odd. For m odd $\theta_m(\tau, 0) \equiv 0$.

Definition

For $\varepsilon \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$ the theta function of the second order is

$$\Theta[\varepsilon](\tau, z) := \theta\begin{bmatrix}\varepsilon \\ 0\end{bmatrix}(2\tau, 2z).$$

- Theta functions of the second order generate $H^0(A_\tau, 2\Theta)$.

- Theta constants $\theta_m(\tau,0)$ are modular forms of weight $1/2$ for a certain finite index normal subgroup $\Gamma(2n,4n)\subset \mathrm{Sp}(2g,\mathbb{Z})$.
- Theta constants of the second order $\Theta[\varepsilon](\tau,0)$ are modular forms of weight $1/2$ for $\Gamma(2,4)$.

Theorem (Igusa, Mumford, Salvati Manni)

For any $n\geq 2$ theta constants embed

$$\mathcal{A}_g(2n,4n):=\mathcal{H}_g/\Gamma(2n,4n)\hookrightarrow \mathbb{P}^{n^{2g}-1}$$

$$\tau\mapsto\left\{\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau)\right\}_{\text{all }\varepsilon,\delta\in\frac{1}{n}\mathbb{Z}^g/\mathbb{Z}^g}$$

- Theta constants of the second order define a generically injective $Th:\mathcal{A}_g(2,4)\to\mathbb{P}^{2^g-1}$ (conjecturally an embedding).

Classical Riemann-Schottky problem

Write the defining equations for

$$\overline{Th(\mathcal{J}_g(2,4))}\subset\overline{Th(\mathcal{A}_g(2,4))}\subset\mathbb{P}^{2^g-1}.$$

g	deg $Th(\mathcal{J}_g(2,4))$		deg $Th(\mathcal{A}_g(2,4))$
1	1	$=$	1
2	1	$=$	1
3	16	$=$	16
4	208896	$=16\cdot$	13056

Theorem (Schottky, Igusa)

The defining equation for $\mathcal{J}_4\subset\mathcal{A}_4$ is

$$F_4:=2^4\sum_{m\in A[2]}\theta_m^{16}(\tau)-\left(\sum_{m\in A[2]}\theta_m^8(\tau)\right)^2$$

Open Problem

Construct all geodesics for the metric on $\mathcal{A}_4$ contained in $\overline{\mathcal{M}_4}$.

In terms of lattice theta functions,

$$\sum \theta_m^{16}(\tau) = \theta_{D_{16}^+}(\tau), \qquad \sum \theta_m^8(\tau) = \theta_{E_8}(\tau).$$

Physics conjecture (Belavin, Knizhnik, D'Hoker-Phong, ...)

The ... $SO(32)$ ... type ... superstring theory ... and ... $E_8 \times E_8$ theory ... are dual, thus ... expectation values of ... are equal ..., so $\theta_{D_{16}^+} \simeq \theta_{E_8 \times E_8}$, and thus

$$F_g := 2^g \sum \theta_m^{16} - \left(\sum \theta_m^8\right)^2$$

vanishes on $\mathcal{J}_g$ for any g (this is true for $g \leq 4$).

Theorem (G.-Salvati Manni)

This conjecture is false for any $g \geq 5$.
In fact the zero locus of F_5 on $\mathcal{M}_5$ is the divisor of trigonal curves.

g	$\deg Th(\mathcal{J}_g(2,4))$	$\deg Th(\mathcal{A}_g(2,4))$
1	1	1
2	1	1
3	16	16
4	208896	13056
5	282654670848	1234714624
6	23303354757572198400	25653961176383488
7	87534047502300588892024209408	197972857997555419746140160

These are the top self-intersection numbers of $\lambda_1/2$ on $\mathcal{M}_g$ and $\mathcal{A}_g$ times the degree of $\mathcal{A}_g(2,4) \to \mathcal{A}_g$. (G., using Faber's algorithm)

Corollary

$Th(\mathcal{J}_g(2,4)) \subset Th(\mathcal{A}_g(2,4))$ is not a complete intersection for $g = 5, 6, 7$. (previously proven by Faber)

Challenge

Write at least one (nice/invariant) modular form vanishing on $\mathcal{J}_5$.

The hyperelliptic Schottky problem

Theorem (Mumford, Poor)

For any g there exist sets of characteristics $S_1, \dots, S_N \subset \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ such that $\tau \in \mathcal{A}_g$ is the period matrix of a hyperelliptic Jacobian ($\tau \in Hyp_g$) if and only if for some $1 \le i \le N$

$$\forall m \qquad \{\theta_m(\tau) = 0 \Longleftrightarrow m \in S_i\}$$

Schottky-Jung approach (H. Farkas-Rauch)

Definition

The Prym variety for an étale double cover $\tilde{C} \to C$ of $C \in \mathcal{M}_g$ (given by a point $\eta \in Jac(C)[2] \setminus \{0\}$) is

$$Prym(C,\eta) := Ker_0(Jac(\tilde{C}) \to Jac(C)) \in \mathcal{A}_{g-1}$$

Denote $\mathcal{P}_g \subset \mathcal{A}_g$ the Prym locus.

Theorem (Schottky-Jung, Farkas-Rauch proportionality)

Let τ be the period matrix of C and let π be the period matrix of the Prym (for the simplest choice of η). Then

$$\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\pi)^2 = const\,\theta\begin{bmatrix}0\,\varepsilon\\ 0\,\delta\end{bmatrix}(\tau)\cdot\theta\begin{bmatrix}0\,\varepsilon\\ 1\,\delta\end{bmatrix}(\tau) \quad \forall\varepsilon,\delta \in \frac{1}{2}\mathbb{Z}^{g-1}/\mathbb{Z}^{g-1}$$

Using this allows us to get some equations for $Th(\mathcal{J}_g(2,4))$ from equations for $Th(\mathcal{A}_{g-1}(2,4))$.

Theorem (van Geemen / Donagi)

The locus $\mathcal{J}_g$ is an irreducible component of the small / big Schottky-Jung locus — the locus obtained by taking the ideal of equations defining $Th(\mathcal{A}_{g-1}(2,4))$ and applying the proportionality for all / for just one double covers(s) η.

Conjecture

$\mathcal{J}_5$ is equal to the "small" (i.e., if we take all η) Schottky-Jung locus in genus 5.

- The locus of intermediate Jacobians of cubic threefolds is contained in the "big" (if we take just one η) Schottky-Jung locus in genus 5.
- For $g \geq 7$, $\overline{\mathcal{P}_{g-1}} \subsetneq \mathcal{A}_{g-1}$, so may have more equations $\Rightarrow$ need to solve the Prym Schottky problem if the above is not enough.

Equations for theta constants: recap

\+ We get explicit algebraic equations for theta constants.

We do get the one defining equation for $\mathcal{J}_4$.

Get 8 conjectural defining equations for $\mathcal{J}_5$ [Accola] that involve lots of combinatorics, unlike the defining equation F_4 for $\mathcal{J}_4$.

— We do not really know $Th(\mathcal{A}_{g-1}(2,4))$ entirely (though we do know many elements of the ideal).

This is so far a "weak" (i.e., up to extra components) solution to the Schottky problem.

Boundary degeneration of Pryms is hard [Alexeev-Birkenhake-Hulek]

Singularities of the theta divisor approach

For $C \in Hyp_g$ have $\dim(\operatorname{Sing}\Theta_{Jac(C)}) = g-3$.
For $C \in \mathcal{M}_g \setminus Hyp_g$ have $\dim(\operatorname{Sing}\Theta_{Jac(C)}) = g-4$.
(By Riemann's theta singularity theorem)

Definition (Andreotti-Mayer loci)

$$N_k := \left\{(A,\Theta) \in \mathcal{A}_g \mid \dim \operatorname{Sing}\Theta \geq k\right\}$$

Theorem (Andreotti-Mayer)

Hyp_g is an irreducible component of N_{g-3}.
$\mathcal{J}_g$ is an irreducible component of N_{g-4}.

Theorem (Debarre)

$\mathcal{P}_g$ is an irreducible component of N_{g-6}.

Andreotti-Mayer divisor N_0

- $N_0 \subsetneq \mathcal{A}_g$ [Andreotti-Mayer]
- N_0 is a divisor in $\mathcal{A}_g$ [Beauville]
- $N_0 = 2N_0' \cup \theta_{\mathrm{null}}$, two irreducible components [Debarre]

Definition (Theta-null divisor)

$$\theta_{\mathrm{null}} := \left\{\tau \mid \prod_{\varepsilon,\delta \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g \,\mathrm{even}} \theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau) = 0\right\}$$
$$= \left\{(A,\Theta) \in \mathcal{A}_g \mid A[2]^{\mathrm{even}} \cap \Theta \neq \emptyset\right\}$$

- $N_1 \subsetneq N_0$, $\operatorname{codim}_{\mathcal{A}_g} N_1 \geq 2$ [Mumford]
- $\operatorname{codim}_{\mathcal{A}_g} N_1 \geq 3$ [Ciliberto-van der Geer]

Conjecture (Beauville, Debarre, ...)

$N_{g-3} = Hyp_g; \quad N_{g-4} \setminus \mathcal{J}_g \subset \theta_{\rm null}$ within $\mathcal{A}_g^{indec}$

Thus interested in $\mathcal{J}_g \cap \theta_{\rm null}$.
For genus 4 have $N_0 = \mathcal{J}_4 \cup \theta_{\rm null}$, so $\mathcal{J}_4 \setminus \theta_{\rm null} = N_0 \setminus \theta_{\rm null}$.

Conjecture (H. Farkas)
Theorem (G.-Salvati Manni; Smith-Varley)

$$\mathcal{J}_4 \cap \theta_{\rm null} = \left\{ \exists m \in A[2]^{\rm even} \quad \theta(\tau, m) = \det_{i,j} \partial_{z_i}\partial_{z_j}\theta(\tau, m) = 0 \right\}$$
$$= \exists m \in A[2]^{\rm even} \cap \Theta;\ TC_m\Theta \text{ has rank} \leq 3 \quad =: \theta^3_{\rm null}$$

Theorem (G.-Salvati Manni, Smith-Varley + Debarre)

$(\mathcal{J}_g \cap \theta_{\rm null}) \subset \theta^3_{\rm null} \subset \theta^{g-1}_{\rm null} \subset (\theta_{\rm null} \cap N_0') \subset \operatorname{Sing} N_0$

More questions on N_k

Note that $\Theta_{A_1 \times A_2} = (\Theta_{A_1} \times A_2) \cup (A_1 \times \Theta_{A_2})$.
Thus $\operatorname{Sing}(\Theta_{A_1 \times A_2}) \supset \Theta_{A_1} \times \Theta_{A_2}$.

Conjecture (Arbarello-De Concini)
Theorem (Ein-Lazarsfeld)

$$N_{g-2} = \mathcal{A}_g^{decomposable}$$

Conjecture (Ciliberto-van der Geer)

$$\operatorname{codim}_{\mathcal{A}_g^{indec}} N_k \geq \frac{(k+1)(k+2)}{2}$$

Question

Is it possible that $N_k = N_{k+1}$ for some k?

Multiplicity of the theta divisor

Theorem (Kollár)

For any $(A,\Theta) \in \mathcal{A}_g$, *any* $z \in A$ *we have* $\mathrm{mult}_z\Theta \le g$.

Theorem (Smith-Varley)

If $\mathrm{mult}_z\Theta = g$, *then* $A = E_1 \times \cdots \times E_g$.

Conjecture

For $A \in \mathcal{A}_g^{indec}$ and any $z \in A$, $\mathrm{mult}_z\Theta \le \left\lfloor \frac{g+1}{2} \right\rfloor$.

- The bound holds and is achieved for Jacobians [Riemann]
- The bound holds and is achieved for Pryms [Mumford, Smith-Varley, Casalaina-Martin]
- Thus the conjecture is true for $g \le 5$ ($\overline{\mathcal{P}_5} = \mathcal{A}_5$)

Andreotti-Mayer approach: recap

\+ Geometric conditions for an abelian variety to be a Jacobian.

Geometric solution in genus 4.

− $\dim \mathrm{Sing}\,\Theta_\tau$ hard to compute for an explicitly given $\tau \in \mathcal{H}_g$.

Only a weak solution (at least so far) in higher genera.

Curves of small homology class

For Jacobians have the Abel-Jacobi curve $C \hookrightarrow Jac(C)$.

Theorem (Matsusaka-Ran)

If $\exists C \subset A$ of "minimal" class $\frac{\Theta^{g-1}}{(g-1)!}$, then $A = Jac(C)$

For Pryms the Abel-Prym curve $\tilde{C} \hookrightarrow Jac(\tilde{C}) \to Prym(\tilde{C} \to C)$.

Theorem (Welters)

If $\exists C \subset A$ of homology class $2\frac{\Theta^{g-1}}{(g-1)!}$, then A is a Prym (or a degeneration, technical details, ...)

Here we start with a curve and solve an easier version of Schottky: *Given $C \subset A$, is $A = Jac(C)$?*

Geometry of the Kummer variety

Definition

The Kummer variety is the image of

$$Kum := |2\Theta| : A_\tau / \pm 1 \hookrightarrow \mathbb{P}^{2^g-1}$$

$$z \to \left\{\Theta[\varepsilon](\tau, z)\right\}_{\text{all } \varepsilon \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g},$$

Trisecant formula (Fay, Gunning)

$\forall p, p_1, p_2, p_3 \in C \subset Jac(C) = Pic^0(C)$ the following are collinear:

$$Kum(p+p_1-p_2-p_3), Kum(p+p_2-p_1-p_3), Kum(p+p_3-p_1-p_2) \quad (*)$$

Theorem (Gunning)

If for some $A \in \mathcal{A}_g^{indec}$ there exist infinitely many p such that $(*)$ (p_i fixed, in general position), then $A \in \mathcal{J}_g$.

This is a solution to the Schottky problem, already given a curve.

“Getting rid” of the points of secancy

“Multi”secant formula (Gunning)

For any $1 \le k \le g$ and for any $p_1, \dots, p_{k+2}, q_1, \dots, q_k \in C \subset Jac(C)$ the $k+2$ points

$$Kum(2p_j + \sum_{i=1}^{k} q_i - \sum_{i=1}^{k+2} p_i), \qquad j = 1 \dots k+2$$

are linearly dependent.

Note $Sym^g C \twoheadrightarrow Jac(C)$, use $k = g$ above. The converse is

Conjecture (Buchstaber-Krichever)
Theorem (G., Pareschi-Popa)

Given $A \in \mathcal{A}_g^{indec}$ and $p_1, \dots, p_{g+2} \in A$ *in general position*, if

$$\forall z \in A \qquad \{Kum(2p_i + z)\}_{i=1\dots g+2} \subset \mathbb{P}^{2^g - 1}$$

are linearly dependent, then $A \in \mathcal{J}_g$.

Trisecant Conjecture (Welters)
Theorem (Krichever)

If $Kum(A)$ has a trisecant, for $A \in \mathcal{A}_g^{indec}$, then $A \in \mathcal{J}_g$.

- No general position assumption: only that the points of secancy are not in $A[2]$, so that $Kum(A)$ is smooth at them.
- Also true for degenerate trisecants, i.e., the existence of a
  - Semidegenerate trisecant: a line tangent to $Kum(A)$ at a point not in $A[2]$ intersecting $Kum(A)$ at another point

  or
  - Flex line: a line tangent to $Kum(A)$ at a point not in $A[2]$ with multiplicity 3

  implies that A is a Jacobian.

Kummer images of Prym varieties

Theorem (Fay, Beauville-Debarre)

For any $p, p_1, p_2, p_3 \in \tilde{C} \to Prym(\tilde{C} \to C)$ the points

$$\begin{array}{ll} Kum(p + p_1 + p_2 + p_3), & Kum(p + p_1 - p_2 - p_3), \\ Kum(p + p_2 - p_1 - p_3), & Kum(p + p_3 - p_1 - p_2) \end{array} \qquad (**)$$

lie on a 2-plane in $\mathbb{P}^{2^g-1}$.

Theorem (Debarre)

If for some $A \in \mathcal{A}_g^{indec}$ there exist infinitely many p such that $(**)$ (p_i fixed and in general position), then $A \in \mathcal{P}_g$.

Example (Beauville-Debarre)

There exists $A \in \mathcal{A}_g \setminus \overline{\mathcal{P}_g}$ such that $Kum(A)$ has a quadrisecant.

Theorem (G.-Krichever)

For $A \in \mathcal{A}_g^{indec}$ and $p, p_1, p_2, p_3 \in A$, if $(**)$, and $(**)$ also holds for $-p, p_1, p_2, p_3$, then $A \in \overline{\mathcal{P}_g}$.

(Characterization by a symmetric pair of quadrisecants)

Secants of the Kummer variety: recap

+ A "strong" solution to Schottky and Prym-Schottky (no extra components).

  Finite amount of data involved, no curves or infinitesimal structure.

− The points of the tri(quadri)secancy enter in the equations, i.e., we do not directly get algebraic equations for theta constants.

Challenge

Use these characterizations to approach Coleman's conjecture, or solve the Torelli problem for Pryms (period map generically injective — conjecturally the non-injectivity is due only to the tetragonal construction), or ...

Theorem (Buser, Sarnak)

The upper bound for the length of the shortest period for Jacobians is (much) less than the upper bound for the length of the shortest period for abelian varieties.

Theorem (Lazarsfeld, also work by Bauer, Nakamaye)

The Seshadri constant for a generic Jacobian is much smaller than for a generic abelian variety.

+ Gives a way to tell that some abelian varieties are *not* Jacobians.
— Does not possibly give a way to show that a given abelian variety *is* a Jacobian, or does it?

Can characterize Hyp_g by the value of their Seshadri constant, if the Γ_{00} conjecture holds [Debarre, Lazarsfeld]

Γ_{00} conjecture

Definition

$$\Gamma_{00} = \{f \in H^0(A, 2\Theta) \mid mult_0 f \geq 4\}$$

Theorem (set-theoretically: Welters, scheme-theoretically: Izadi)

For $g \geq 5$ on $Jac(C)$ we have $Bs(\Gamma_{00}) = C - C$.

Conjecture (van Geemen-van der Geer)

For $A \in \mathcal{A}_g^{indec}$ if $Bs(\Gamma_{00}) \neq \{0\}$, then $A \in \mathcal{J}_g$.

- Holds for $g = 4$. [Izadi]
- Holds for a generic Prym for $g \geq 8$. [Izadi]
- Holds for a generic abelian variety for $g = 5$ or $g \geq 14$. [Beauville-Debarre-Donagi-van der Geer]

- Even functions $\Theta[\varepsilon](\tau, z)$ generate $H^0(A, 2\Theta)$.

Thus

$$z \in Bs(\Gamma_{00})$$
$$\Updownarrow$$
$$Kum(z) \in \langle Kum(0), \partial_{z_i}\partial_{z_j} Kum(0)\rangle_{\text{linear span}}$$
$$\Updownarrow$$
$$Kum(z) = cKum(0) + \sum c_{ij}\partial_{z_i}\partial_{z_j} Kum(0) \qquad (\dagger)$$

for some $c, c_{ij} \in \mathbb{C}$

- Similar to a semidegenerate trisecant tangent at $Kum(0)$.
- For $p, q \in C \subset Jac(C)$ in fact $\text{rk}(c_{ij}) = 1$ [∼Frobenius]

Theorem (G.)

For $A \in \mathcal{A}_g^{indec}$ if $(\dagger)$ holds with $\text{rk}(c_{ij}) = 1$, then $A \in \mathcal{J}_g$.

Idea (Muñoz-Porras)

If Γ_{00} conjecture holds, then $\mathcal{J}_g$ =small Schottky-Jung locus (methods to prove this by degenerating to the boundary).

Proofs

For the results on Schottky's form and theta divisors:

Theta functions are not a spectator sport...

--- Lipman Bers

For the characterization of Pryms by pairs of quadrisecants:
Use integrable systems.
Dubrovin, Krichever, Novikov, Arbarello, De Concini, Shiota, Mulase, Marini, Muñoz Porras, Plaza Martin, ...

...and neither are integrable systems.

- Kadomtsev-Petviashvili (KP) equation is the condition for the existence of a 1-jet of a family of degenerate trisecants (flex lines of the Kummer).
- KP hierarchy of PDEs is the hierarchy of the conditions for the existence of n-jets of a family of flex lines, for each $n \in \mathbb{N}$.
- The existence of an n-jet of a family of flexes for any n gives a formal one-dimensional family of flexes.
- Such a formal family comes from an actual geometric family.
- Thus if the KP hierarchy is satisfied by the theta function, the abelian variety is a Jacobian.
- Shiota proved that the KP equation suffices to recover the hierarchy.
- Krichever showed that the obstruction for extending one flex to a family of flexes vanishes in Taylor series.
- For Pryms, G.-Krichever needed a new hierarchy, etc.