Fall 2018 MAT 320
Practice Final problems

Problem 1. Suppose a sequence (s_n) of real numbers is such that for any n we have $|s_n - s_{n+2}| < 2^{-n}$. Can s_n be divergent?

Problem 2. Construct a sequence (s_n) such that the set S of its subsequential limits is equal to $S = \{0, 1, 2, 3\}$

Problem 3. Let (s_n) be the sequence of partial sums of a series $\sum a_n$. If the series diverges, can (s_n) be

a) a bounded sequence?

b) a bounded increasing sequence?

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that its square $f^2(x)$ is a function continuous everywhere. Does it follow that f is continuous everywhere?

Problem 5. a) Suppose a function $f : \mathbb{R} \to \mathbb{R}$ is such that $\forall \varepsilon > 0$, $\exists \delta > 0$, such that for any $x \in \mathbb{R}$, $|x| < \delta$ implies $|f(x)| < x\varepsilon$. Does this imply that f is continuous at 0?

b) Answer the same question, if the last inequality is replaced with $|f(x)| < \frac{\varepsilon}{x}$.

[in both cases, either prove that f must be continuous at 0, or give an example of f that is not continuous, and prove that it is not continuous, and satisfies the conditions]

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let $g : \mathbb{R} \to \mathbb{R}$ be some function such that $\lim_{x \to a} (f \circ g)(x) = (f \circ g)(a)$. Does it necessarily follow that $\lim_{x \to a} g(x) = g(a)$?

Problem 7. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function differentiable at a. Compute the limit $\lim_{h \to 0} \frac{f(a+h) - f(a-h)}{h}$, in terms of $f'(a)$.

Problem 8. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} 1 - x & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}.$$

Is f continuous on $[-1, 1]$? differentiable on $[-1, 1]$? Does it satisfy the conclusions of the Mean Value Theorem and the Intermediate Value Theorem on $[-1, 1]$?

Problem 9. Suppose f, g are continuous functions on $[0, 1]$ that are differentiable on $(0, 1)$, such that $f(0) = g(0) = 0$, $g(1) = 1$, and for any $x \in (0, 1)$ we have $f'(x) \leq g'(x) \cdot g(x)$. Prove that $f(1) \leq \frac{1}{2}$.

Problem 10. Suppose a sequence of differentiable functions $f_n : \mathbb{R} \to [0, 1]$ converges pointwise to the zero function. Does it follow that the derivatives f_n' converge pointwise to the zero function?

Problem 11. Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}^2$ such that the preimage of the closed unit disk $x^2 + y^2 \leq 1$ is the closed interval $[-1, 1]$? the open interval $(-1, 1)$?

Problem 12. Prove the theorem stating that in any metric space any compact set is closed.

Problem 13. Let S_∞ be the set of continuous functions $f : [0, 1] \to [0, 1]$, with the distance $d_\infty(f, g) = \sup |f(x) - g(x)|$, and let S_1 be the (same) set of continuous functions on $[0, 1]$, but with the distance

$$d_1(f, g) = \int_0^1 |f(x) - g(x)| \, dx.$$

Let $F : S_\infty \to S_1$ be the identity map taking any function f to itself. Is F continuous? is F^{-1} continuous? (You are allowed to use any properties of the integral you want without proof)

Problem 14. Show that the metric space S_∞ in the previous problem is connected. Without the previous problem, does it automatically follow whether S_1 is connected? Given the result of the previous problem, does it follow that S_1 is also connected?

The actual final will have 8 problems, and a bonus question on dense/nowhere dense sets/Baire category